1. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and π^0, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2) \cong \mathrm{Spin}(4)$ which would be exact if the two lightest quark flavors u and d were exactly massless; in reality, the current quark masses m_u and m_d do not exactly vanish but are small enough to be treated as a perturbation. Exact or approximate, the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken $\mathrm{Spin}(4)/\mathrm{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and π^0.

   QCD is a rather complicated theory, so it is often convenient to describe the physics of the spontaneously broken chiral symmetry in terms of a simpler effective theory such as the linear sigma model. This model has 4 real scalar fields; in terms of the unbroken isospin symmetry, we have an isosinglet $\sigma(x)$ and an isotriplet $\underset{\sim}{\pi}(x)$ comprising $\pi^1(x)$, $\pi^2(x)$ and $\pi^3(x)$ (or equivalently, $\pi^0(x) \equiv \pi^3(x)$ and $\pi^\pm(x) \equiv \left(\pi^1(x) \pm i\pi^2(x)\right)/\sqrt{2}$). The Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;+\; \tfrac{1}{2}(\partial_\mu\underset{\sim}{\pi})^2 \;-\; \frac{\lambda}{8}\left(\sigma^2 + \underset{\sim}{\pi}^2 - f^2\right)^2 \;+\; \beta\sigma \tag{1}$$

   is invariant under the $SO(4)$ rotations of the four fields, except for the last term which we take to be very small. (In QCD $\beta \sim \frac{m_u+m_d}{2f}\left\langle\overline{\Psi}\Psi\right\rangle$ which is indeed very small because the u and d quarks are very light.)

   In class, we discussed this theory for $\beta = 0$ and showed that it has $SO(4)$ spontaneously broken to $SO(3)$ and hence 3 massless Goldstone bosons. In this exercise, we let $\beta > 0$ but $\beta \ll \lambda f^3$ to show how this leads to massive but light pions.

(a) Show that the scalar potential of the linear sigma model with $\beta > 0$ has a unique minimum at

$$\langle \underset{\sim}{\pi} \rangle = 0 \quad \text{and} \quad \langle \sigma \rangle = f + \frac{\beta}{\lambda f^2} + O(\beta^2). \tag{2}$$

(b) Expand the fields around this minimum and show that the pions are light while the σ particle is much heavier. Specifically, $M_\pi^2 \approx (\beta/f)$ while $M_\sigma^2 \approx \lambda f^2$.

2. The rest of this homework is about the Bogolyubov transform and the superfluid helium. Let us start with some kind of annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^\dagger$ be satisfying the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'} . \tag{3}$$

Let us define new operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}^\dagger$ according to

$$\hat{b}_{\mathbf{k}} = \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}} + \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}}^\dagger , \quad \hat{b}_{\mathbf{k}}^\dagger = \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}}^\dagger + \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}} \tag{4}$$

for some arbitrary real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$.

(a) Show that the $\hat{b}_{\mathbf{k}}$ and the $\hat{b}_{\mathbf{k}}^\dagger$ satisfy the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and the $\hat{a}_{\mathbf{k}}^\dagger$.

The Bogolyubov transform — replacing the 'original' creation and annihilation operators $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ with the 'transformed' operators $\hat{b}_{\mathbf{k}}^\dagger$ and $\hat{b}_{\mathbf{k}}$ — is useful for diagonalizing quadratic Hamiltonians of the form

$$\hat{H} = \sum_{\mathbf{k}} A_{\mathbf{k}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \tfrac{1}{2} \sum_{\mathbf{k}} B_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger \right) \tag{5}$$

where for all momenta $\mathbf{k}$, $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$.

(b) Show that for a suitable choice of the $t_{\mathbf{k}}$ parameters,

$$\hat{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}} + \text{const} \qquad \text{where } \omega_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - B_{\mathbf{k}}^2} . \tag{6}$$

Now consider the quantum field theory of superfluid helium. As discussed in class, we start with a semi-classical ground state with a symmetry-breaking expectation value $\left\langle \hat{\Psi}(\mathbf{x}) \right\rangle \equiv \sqrt{n} = \sqrt{\mu/\lambda}$ and shift the quantum fields according to

$$\hat{\Psi}(\mathbf{x}) \;=\; \sqrt{n} \,+\, \hat{\varphi}(\mathbf{x}), \qquad \hat{\Psi}^\dagger(\mathbf{x}) \;=\; \sqrt{n} \,+\, \hat{\varphi}^\dagger(\mathbf{x}). \tag{7}$$

In terms of the shifted fields the free energy operator becomes

$$\hat{H} - \mu\hat{N} \;=\; \text{const} \,+\, \hat{H}_{\text{free}} \,+\, \hat{H}_{\text{int}}$$

where

$$\hat{H}_{\text{free}} \;=\; \int d^3\mathbf{x} \left\{ \frac{1}{2M}\, \nabla\hat{\varphi}^\dagger \cdot \nabla\hat{\varphi} \;+\; \frac{\lambda n}{2} \left(\hat{\varphi}^\dagger\hat{\varphi}^\dagger + 2\hat{\varphi}^\dagger\hat{\varphi} + \hat{\varphi}\hat{\varphi} \right) \right\} \tag{8}$$

is quadratic with respect to the shifted fields while $\hat{H}_{\text{int}}$ comprises the cubic and the quartic terms.

(c) The momentum modes of the shifted fields are shifted creation and annihilation operators $\tilde{a}^\dagger_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} - \sqrt{N}\delta_{\mathbf{k},\mathbf{0}}$ and $\tilde{a}_{\mathbf{k}} = \hat{a}_{\mathbf{k}} - \sqrt{N}\delta_{\mathbf{k},\mathbf{0}}$. Apply Bogolyubov transform to the shifted operators and re-write the free Hamiltonian (8) as

$$\hat{H}_{\text{free}} \;=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;+\; \text{const} \tag{9}$$

where

$$\omega_{\mathbf{k}} \;=\; |\mathbf{k}| \times \sqrt{\frac{\lambda n}{M} \,+\, \frac{\mathbf{k}^2}{4M^2}}. \tag{10}$$

The ground state of the free Hamiltonian (9) is the state $|\Omega_2\rangle$ annihilated by all the $\hat{b}_{\mathbf{k}}$ operators. To construct this state, we start with the semi-classical ground state — the coherent state $|\text{coh}\rangle$ in which $\hat{\Psi}(\mathbf{x})\,|\text{coh}\rangle \equiv \sqrt{n}\,|\text{coh}\rangle$ and therefore $\tilde{a}_{\mathbf{k}}\,|\text{coh}\rangle = 0$ for all $\mathbf{k}$ — and then act with a unitary operator $e^{\hat{F}}$ according to

$$|\Omega_2\rangle \;=\; e^{\hat{F}}\,|\text{coh}\rangle \quad \text{where} \quad \hat{F} \;=\; \sum_{\mathbf{k}} \frac{t_{\mathbf{k}}}{2} \left(\tilde{a}^\dagger_{\mathbf{k}} \tilde{a}^\dagger_{-\mathbf{k}} \,-\, \tilde{a}_{-\mathbf{k}} \tilde{a}_{\mathbf{k}} \right) \;=\; -\hat{F}^\dagger \tag{11}$$

(∗) Optional exercise: Show that $\hat{b}_{\mathbf{k}} = e^{\hat{F}} \tilde{a}_{\mathbf{k}} e^{-\hat{F}}$, and hence $\hat{b}_{\mathbf{k}}\,|\Omega_2\rangle = 0$ for all $\mathbf{k}$ as well as automatic bosonic commutation relations for the $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ operators.

The excited states of the free Hamiltonian (9) can be constructed by applying the $\hat{b}^\dagger_{\mathbf{k}}$ operators to the ground state $|\Omega_2\rangle$. Thus, one can say that *the* $\hat{b}^\dagger_{\mathbf{k}}$ *operators create quasi-particles and the the* $\hat{b}_{\mathbf{k}}$ *operators annihilate them*; from this point of view, the $|\Omega_2\rangle$ ground state is the *quasi-particle vacuum*.

(d) Show that the quasi-particle created by the $\hat{b}^\dagger_{\mathbf{k}}$ operator and annihilated by the $\hat{b}_{\mathbf{k}}$ operator has a definite mechanical momentum $\mathbf{k}$, and that

$$\hat{\mathbf{P}}_{\text{tot}} = \sum_{\mathbf{k}} \mathbf{k}\, \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}. \tag{12}$$

On the other hand, the quasi-particles do not have well-defined atomic numbers. This is related to the spontaneous breakdown of the phase symmetry, which is generated by the atom number operator $\hat{N}$. Physically, the quasi-particles interpolate between phonons in the superfluid (for small $\mathbf{k}$) and atoms knocked out of the Bose condensate (for large $\mathbf{k}$) — note the appropriate limits of the dispersion relation (10).

(e) Check that for large momenta $\hat{b}^\dagger_{\mathbf{k}} \approx \hat{a}^\dagger_{\mathbf{k}}$ and therefore the quasi-particle is approximately an atom, while for small momenta $\hat{b}^\dagger_{\mathbf{k}} \approx (\text{coeff}) \times (\hat{a}^\dagger_{\mathbf{k}} + \hat{a}_{\mathbf{k}})$ and therefore the quasiparticle is approximately an atom.

Actually, in the real helium with a finite-range interatomic potential $V_2(\mathbf{x} - \mathbf{y})$, the dispersion relation is a bit more complicated than eq. (10) — *e.g.*, there is a so-called 'roton dip' at intermediate values of the quasiparticle momenta $\mathbf{k}$ — but the small–$\mathbf{k}$ and the large–$\mathbf{k}$ limits work exactly as in this exercise. In particular, there is a positive lower bound on quasi-particle phase velocities: $\forall\mathbf{k},\ \omega_{\mathbf{k}} \geq v_c|\mathbf{k}|$. This fact plays a crucial role in superfluidity.

(f) Consider the superfluid in a state of uniform motion with velocity $\mathbf{v}$. Use Galilean invariance to argue that quasi-particles in moving Helium are governed by the

$$\hat{H}'_{\text{free}} = \hat{H}_{\text{free}} + \mathbf{v} \cdot \hat{\mathbf{P}} = \sum_{\mathbf{k}} (\omega_{\mathbf{k}} + \mathbf{v}\mathbf{k})\, \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\,. \tag{13}$$

Therefore, as long as $|\mathbf{v}| < v_c$, all excitations have positive energies, hence there is no spontaneous decay of the flowing "ground" state and no energy dissipation! This is the physical origin of superfluidity.

On the other hand, when the Helium flows too fast, $|\mathbf{v}| > v_0$, some quasiparticle modes acquire negative energies, which leads to spontaneous quasiparticle production, hence energy dissipation and loss of superfluidity.

The *critical velocity* v_c is governed by the dispersion relation for the quasi-particles: $v_c = \min(\omega_{\mathbf{k}}/|\mathbf{k}|)$. For the superfluid, $v_c > 0$. In comparison, the ideal gas has $\omega_{\mathbf{k}} = \mathbf{k}^2/2m$, thus $v_c = 0$ and no superfluidity at any velocity.

Actually, under most experimental conditions, there is an additional mechanism for losing superfluidity beyond a much smaller critical velocity than the v_c obtaining from the microscopic theory. Specifically, turbulence leads to spontaneous generations of vortex rings, which move much slower than the quasi-particles and hence quench superfluidity at much slower speeds. In very thin capillaries however, the vortex rings do not form because of size limitations and the superfluidity persists until the microscopic critical velocity v_c.