1. Back in homework#2 we developed Hamiltonian formalism for a massive vector field $A^\mu(x)$. Upon quantization, the 3–vector field $\mathbf{A}(x)$ and its canonical conjugate $-\mathbf{E}(x)$ become quantum fields subject to equal-time commutation relations

$$[\hat{A}^i(\mathbf{x}),\hat{A}^j(\mathbf{y})] \;=\; 0, \quad [\hat{E}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})] \;=\; 0, \quad [\hat{A}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})] \;=\; -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1}$$

($\hbar=1, c=1$ units) governed by the free Hamiltonian

$$\hat{H} \;=\; \int d^3\mathbf{x} \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 \;+\; \frac{(\nabla\cdot\hat{\mathbf{E}})^2}{2m^2} \;+\; \tfrac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 \;+\; \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 \right) \tag{2}$$

(we assume $J^\mu=0$). For the non-dynamical A^0 field, its time-independent equation of motion becomes an operatorial identity

$$\hat{A}^0(x) \;=\; -\frac{\nabla\cdot\hat{\mathbf{E}}(x)}{m^2}\,. \tag{3}$$

The purpose of the present exercise is to expand fields in terms of creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$ where λ labels three different polarization states of a vector particle (spin = 1). Generally, bases for polarization states correspond to $\mathbf{k}$–dependent complex bases $\mathbf{e}_\lambda(\mathbf{k})$ for ordinary 3–vectors,

$$\mathbf{e}_\lambda(\mathbf{k})\cdot\mathbf{e}^*_{\lambda'}(\mathbf{k}) \;=\; \delta_{\lambda,\lambda'} \tag{4}$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k}\times$, namely

$$i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k}) \;=\; \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \qquad \lambda=-1,0,+1. \tag{5}$$

By convention, the overall phases of the helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) \;=\; \frac{\mathbf{k}}{|\mathbf{k}|}, \quad \mathbf{e}^*_\lambda(\mathbf{k}) \;=\; (-1)^\lambda\mathbf{e}_{-\lambda}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) \;=\; -\mathbf{e}^*_\lambda(+\mathbf{k}). \tag{6}$$

Combining Fourier transform with a basis decomposition, we have

$$\hat{\mathbf{A}}(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{kx}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda}\,, \qquad \hat{A}_{\mathbf{k},\lambda} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{kx}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}(\mathbf{x}) \tag{7}$$

and ditto for the $\hat{\mathbf{E}}(\mathbf{x})$ field and its $\hat{E}_{\mathbf{k},\lambda}$ modes.

(a) Show that $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$, and derive the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ operators.

(b) Show that

$$\hat{H} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\left(\frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;+\; \frac{\omega^2_{\mathbf{k}}}{2C_{\mathbf{k},\lambda}}\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right) \tag{8}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2+m^2}$ and $C_{\mathbf{k},\lambda} = 1+\delta_{\lambda,0}(\mathbf{k}^2/m^2)$.

(c) Define creation and annihilation operators according to

$$\hat{a}_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} \;-\; iC_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}\,, \qquad \hat{a}^\dagger_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}\hat{A}^\dagger_{\mathbf{k},\lambda} \;+\; iC_{\mathbf{k},\lambda}\hat{E}^\dagger_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}} \tag{9}$$

and verify that they satisfy (relativistically-normalized) equal-time bosonic commutation relations.

(d) Show that

$$\hat{H} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda \omega_{\mathbf{k}}\,\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \text{const.} \tag{10}$$

(e) Next, consider the time dependence of the free vector field and show that

$$\hat{\mathbf{A}}(\mathbf{x},t) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_\lambda\sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_\lambda(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}\mathbf{e}^*_\lambda(\mathbf{k})\,\hat{a}^\dagger_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{11}$$

(f) Write down a similar formula for the $\hat{A}^0(\mathbf{x},t)$ (use eq. (3)). Together with the previous result, you should get

$$\hat{A}_\mu(x) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \sum_\lambda \Big(e^{-ikx} f_\mu(\mathbf{k},\lambda)\, \hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx} f^*_\mu(\mathbf{k},\lambda)\, \hat{a}^\dagger_{\mathbf{k},\lambda}(0) \Big)_{k^0=+\omega_{\mathbf{k}}} \tag{12}$$

where

$$f^\mu(\mathbf{k},\lambda) \;=\; \begin{cases} \big(0, \mathbf{e}_\lambda(\mathbf{k})\big) & \text{for } \lambda = \pm 1, \\ \Big(\frac{|\mathbf{k}|}{m}, \frac{\omega_{\mathbf{k}}}{m}\frac{\mathbf{k}}{|\mathbf{k}|}\Big) & \text{for } \lambda = 0. \end{cases} \tag{13}$$

Note that the 4–vectors $f^\mu(\mathbf{k},\lambda)$ are basically the purely-spatial vectors $\mathbf{e}_\lambda(\mathbf{k})$ Lorentz-boosted into the moving particle's frame. In particular, for all $(\mathbf{k},\lambda)$, $f^\mu f^*_\mu = -1$ and $f^\mu k_\mu = 0$.

(g) Finally, verify that the vector field (12) satisfies the free equations of motion $\partial_\mu \hat{A}^\mu(x) = 0$ and $(\partial^2 + m^2)\hat{A}^\mu(x) = 0$.

2. Now consider the Feynman propagator for the massive vector field.

(a) First, a lemma: Show that

$$\sum_\lambda f^\mu(\mathbf{k},\lambda) f^{\nu *}(\mathbf{k},\lambda) \;=\; -g^{\mu\nu} \;+\; \frac{k^\mu k^\nu}{m^2}\,. \tag{14}$$

(b) Next, show that

$$\begin{aligned} \langle 0|\, \hat{A}^\mu(x)\hat{A}^\nu(y)\, |0\rangle \;&=\; \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{2\omega_{\mathbf{k}}} \left[\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik(x-y)} \right]_{k^0=+\omega_{\mathbf{k}}} \\ &=\; \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) D(x-y). \end{aligned} \tag{15}$$

(c) Finally, the Feynman propagator: Show that

$$\begin{aligned} G_F^{\mu\nu} \;\equiv\; \langle 0|\, \mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y)\, |0\rangle \;&=\; \left(-g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{m^2} \right) G_F(x-y) \\ &=\; \int \frac{d^4\mathbf{k}}{(2\pi)^4} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i0} \end{aligned} \tag{16}$$

where

$$\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{17}$$

For the explanation of the $\mathbf{T}^*$ modification of the time-ordered product of vector fields, please see *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.

3. Finally, an exercise in Dirac's γ matrices. Assume the anti-commutation relations

$$\gamma^\mu\gamma^\nu \;+\; \gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu} \tag{18}$$

and the definition

$$S^{\mu\nu} \;=\; -S^{\nu\mu} \;\stackrel{\text{def}}{=}\; \tfrac{1}{4i}[\gamma^\mu,\gamma^\nu] \tag{19}$$

but do not assume the specific form of the 4×4 matrices γ^μ and $S^{\mu\nu}$.

(a) Show that

$$\left[S^{\kappa\lambda},\gamma^\mu\right] \;=\; -ig^{\lambda\mu}\gamma^\kappa \;+\; ig^{\kappa\mu}\gamma^\nu \tag{20}$$

and

$$\left[S^{\kappa\lambda},S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;-\; ig^{\kappa\mu}S^{\lambda\nu} \;+\; ig^{\kappa\nu}S^{\lambda\mu}. \tag{21}$$

(b) Calculate $\{\gamma^\rho,\gamma^\lambda\gamma^\mu\gamma^\nu\}$, $[\gamma^\rho,\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$ and $[S^{\rho\sigma},\gamma^\lambda\gamma^\mu\gamma^\nu]$.

(c) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$ and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$. Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

Continuous Lorentz transforms obtain from integrating infinite sequences of infinitesimal transforms $X'^\mu = X^\mu + \epsilon\Theta^\mu_{\ \nu}X^\nu$ where $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$. Altogether, a finite continuous transform acts as $X'^\mu = L^\mu_{\ \nu}X^\nu$ where

$$L \;=\; \exp(\Theta), \quad i.e., \quad L^\mu_{\ \nu} = \delta^\mu_\nu \;+\; \Theta^\mu_{\ \nu} \;+\; \tfrac{1}{2}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \nu} \;+\; \tfrac{1}{6}\Theta^\mu_{\ \kappa}\Theta^\kappa_{\ \lambda}\Theta^\lambda_{\ \nu} \;+\; \cdots. \tag{22}$$

(d) Let L be a Lorentz transform of the form (22), and let $M(L) = \exp\left(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}\right)$. Show that $M^{-1}(L)\gamma^\mu M(L) = L^\mu_{\ \nu}\gamma^\nu$.