0. First of all, finish problem #3 from the previous homework set #5. It is due by Thursday, October 7.

1. The first problem of this set concerns finite representations of the continuous Lorentz symmetry $SO^+(3,1)$, or rather its double cover $\mathrm{Spin}(3,1) \cong SL(2,\mathbf{C})$. Let us define

$$\hat{\mathbf{J}}_\pm \;=\; \tfrac{1}{2}\big(\hat{\mathbf{J}} \pm i\hat{\mathbf{K}}\big). \tag{1}$$

where $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$ are generators of the Lorentz boosts and $\hat{J}^i$ are generators of the space rotations.

(a) Show that the $\hat{\mathbf{J}}_+$ and the $\hat{\mathbf{J}}_-$ commute with each other and that each satisfies the commutations relations of an angular momentum, $[\hat{J}^k_\pm, \hat{J}^\ell_\pm] = i\epsilon^{k\ell m}\hat{J}^m_\pm$.

The "angular momentum" $\hat{\mathbf{J}}_+$ is non-hermitian and hence its finite irreducible representations are non-unitary analytic continuations of the spin–j representations of a hermitian $\hat{\mathbf{J}}$. The same is true for the $\hat{\mathbf{J}}_- = \hat{\mathbf{J}}_+^\dagger$. Thus altogether, the finite irreducible representations of the Lorentz algebra are specified by two integer or half-integer 'spins' j_+ and j_-.

The simplest non-trivial representations of the Lorentz algebra are $(j_+ = \frac{1}{2}, j_- = 0)$ — the left-handed Weyl spinor where $\hat{\mathbf{J}}$ acts as $\frac{1}{2}\boldsymbol{\sigma}$ and $\hat{\mathbf{K}}$ as $-\frac{i}{2}\boldsymbol{\sigma}$, and $(j_+ = 0, j_- = \frac{1}{2})$ — the right-handed Weyl spinor where $\hat{\mathbf{J}}$ also acts as $\frac{1}{2}\boldsymbol{\sigma}$ but $\hat{\mathbf{K}}$ acts as $+\frac{i}{2}\boldsymbol{\sigma}$. Together, the two Weyl spinors comprise the Dirac spinor. From the $SL(2,\mathbf{C})$ point if view, the left-handed Weyl spinor is the doublet representation $\mathbf{2}$ which defines the $SL(2,\mathbf{C})$ group while the right-handed Weyl spinor is the conjugate doublet $\bar{\mathbf{2}}$. As discussed in class, the Weyl spinors transform according to

$$\psi^L_\alpha \;\mapsto\; M_\alpha^{\ \beta}\psi^L_\beta \qquad \text{and} \qquad (\sigma_2\psi^R)_{\dot\alpha} \;\mapsto\; M^{*\dot\beta}_{\dot\alpha}(\sigma_2\psi^R)_{\dot\beta} \tag{2}$$

where $M \equiv M_L$ and $\sigma_2 M^* \sigma_2 = M_R$. Note the notations: the un-dotted indices from the beginning of the Greek alphabet for the left-handed spinors, and the dotted indices for the right-handed spinors.

A generic (j_+, j_-) representation of the Lorentz algebra becomes in the $SL(2, \mathbf{C})$ terms a tensor $\Phi_{\alpha_1 \ldots \alpha_{(2j_+)}, \dot{\gamma}_1 \ldots \dot{\gamma}_{(2j_-)}}$, totally symmetric in its $2j_+$ un-dotted indices $\alpha_1, \ldots, \alpha_{(2j_+)}$ and separately totally symmetric in its $2j_-$ dotted indices $\dot{\gamma}_1, \ldots, \dot{\gamma}_{(2j_-)}$; it transforms according to

$$\Phi_{\alpha_1 \ldots \alpha_{(2j_+)}, \dot{\gamma}_1 \ldots \dot{\gamma}_{(2j_-)}} \;\mapsto\; M_{\alpha_1}^{\;\beta_1} \cdots M_{\alpha_{(2j_+)}}^{\;\beta_{(2j_+)}}\; M^{*\,\dot{\delta}_1}_{\dot{\gamma}_1} \cdots U^{*\,\dot{\delta}_{(2j_-)}}_{\dot{\gamma}_{(2j_-)}}\; \Phi_{\beta_1 \ldots \beta_{(2j_+)}, \dot{\delta}_1 \ldots \dot{\delta}_{(2j_-)}}\,. \tag{3}$$

The vector representation of the Lorentz group has $j_+ = j_- = \frac{1}{2}$. To cast the action of the Lorentz group in $SL(2, \mathbf{C})$ terms (3), we define a 2×2 matrix

$$X_\mu \sigma^\mu \;\equiv\; X_0 \;-\; \mathbf{X} \cdot \boldsymbol{\sigma} \tag{4}$$

where σ^0 is a unit 2×2 matrix while σ^1, σ^2 and σ^3 are the Pauli matrices. In $SL(2, \mathbf{C})$ terms, this matrix has one dotted and one un-dotted index,

$$X_{\alpha\dot{\gamma}} \;=\; X_\mu \sigma^\mu_{\alpha\dot{\gamma}} \;=\; X_0 \delta_{\alpha\dot{\gamma}} \;-\; \mathbf{X} \cdot \boldsymbol{\sigma}_{\alpha\dot{\gamma}}\,, \tag{5}$$

thus it transforms under the $SL(2, \mathbf{C})$ as a $(\frac{1}{2}, \frac{1}{2})$ bi-spinor,

$$X'_{\alpha\dot{\gamma}} \;=\; M_\alpha^{\;\beta} M^{*\dot{\delta}}_{\dot{\gamma}} X_{\gamma\dot{\delta}}\,, \tag{6}$$

or in matrix form

$$X'_\mu \sigma^\mu \;=\; M (X_\mu \sigma^\mu) M^\dagger. \tag{7}$$

(b) Show that for any $SL(2, \mathbf{C})$ matrix M, eq. (7) defines an orthochronous Lorentz transform $X'_\mu = L_\mu^{\;\nu}(M) X_\nu$. (Hint: prove and use $\det(X_\mu \sigma^\mu) = X^2 \equiv X_\mu X^\mu$).

  ∗ For extra challenge, show that L is proper, *i.e.* $\det(L) = +1$.

(c) Verify the group law, $L(M_2 M_1) = L(M_2) L(M_1)$.

(d) Verify explicitly that for $M = \exp\bigl(-\frac{i}{2}\theta\, \mathbf{n} \cdot \boldsymbol{\sigma}\bigr)$, $L(M)$ is a rotation by angle θ around axis $\mathbf{n}$, while for $M = \exp\bigl(-\frac{1}{2} r\, \mathbf{n} \cdot \boldsymbol{\sigma}\bigr)$, $L(M)$ is a boost of rapidity r ($\beta = \tanh r$, $\gamma = \cosh r$) in the direction $\mathbf{n}$.

In general, any (j_+, j_-) multiplet of the $SL(2,\mathbf{C})$ with integer net spin $j_+ + j_-$ is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the $(1,1)$ multiplet is equivalent to a symmetric, traceless 2–index tensor $T^{\mu\nu} = T^{\nu\mu}$, $T^{\mu}_{\mu} = 0$. For $j_+ \neq j_-$ the representation is complex, but one can make a real tensor by combining two multiplets with opposite j_+ and j_-, for example the $(1,0)$ and $(0,1)$ multiplets are together equivalent to an antisymmetric 2–index tensor $F^{\mu\nu} = -F^{\nu\mu}$.

(e) Verify the above examples.
Hint: For any angular momentum $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$.

The $SL(2,\mathbf{C})$ multiplets with half-integer $j_+ + j_-$ are equivalent to Lorentz spinors or spin-tensors which carry one Weyl index as well as 0, 1 or more 4–vector indices and transform according to

$$\psi_{\alpha}^{\mu,\ldots,\nu} \;\mapsto\; M_{\alpha}^{\ \beta}(L) L^{\mu}_{\ \kappa} \cdots L^{\nu}_{\ \lambda}\, \psi_{\beta}^{\kappa,\ldots,\lambda} \qquad \text{or} \qquad \psi_{\dot\alpha}^{\mu,\ldots,\nu} \;\mapsto\; M_{\dot\alpha}^{*\dot\beta}(L) L^{\mu}_{\ \kappa} \cdots L^{\nu}_{\ \lambda}\, \psi_{\dot\beta}^{\kappa,\ldots,\lambda}. \tag{8}$$

(f) Show that the $(1, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ multiplets are together equivalent to the Rarita–Schwinger spin-vector Ψ^{μ}_{a} which has one Dirac index a and one 4–vector index μ and satisfies a Lorentz-covariant constraint $\gamma_\mu \Psi^\mu = 0$.

2. Now consider an im-proper Lorentz symmetry, namely the *parity* symmetry AKA reflection of space

$$P^{\mu}_{\nu} \;=\; \left\{ \begin{array}{ll} +1 & \text{for } \mu = \nu = 0, \\ -1 & \text{for } \mu = \nu = 1,2,3, \\ 0 & \text{for } \mu \neq \nu, \end{array} \right\} \qquad \text{thus} \quad P(t,\mathbf{x}) \;=\; (+t, -\mathbf{x}); \tag{9}$$

note $P^2 = 1$.

In the Fock space, parity is represented by a unitary operator $\hat{\mathcal{P}}$; by the group law, $\hat{\mathcal{P}}^2 = 1$ and hence $\hat{\mathcal{P}}^\dagger = \hat{\mathcal{P}}^{-1} = \hat{\mathcal{P}}$.

(a) Use group law to show that $\hat{\mathcal{P}}$ commutes with the angular momenta $\hat{J}^i$ but anti-commutes with the boost generators $\hat{K}^i$, then use these commutation relations to show that $\hat{\mathcal{P}}$ acting on the quantum fields must interchange the j_+ and j_- quantum numbers of the field components. For example, parity must turn left-handed Weyl spinors into right-handed Weyl spinors and vice verse, thus $\hat{\mathcal{P}}(\frac{1}{2},0)\hat{\mathcal{P}} = (0,\frac{1}{2})$ and $\hat{\mathcal{P}}(0,\frac{1}{2})\hat{\mathcal{P}} = (\frac{1}{2},0)$.

(b) A Dirac spinor field transforms under parity according to

$$\hat{\mathcal{P}}\,\hat{\Psi}(\mathbf{x},t)\,\hat{\mathcal{P}} \;\equiv\; \hat{\Psi}'(\mathbf{x},t) \;=\; \pm\gamma^0\,\hat{\Psi}(-\mathbf{x},t) \tag{10}$$

where the overall $\pm$ sign is the *intrinsic parity* of a particular Dirac field.

Verify that the Dirac equation is covariant under this transformation and that the Dirac action $\int d^4x\mathcal{L}_{\text{Dirac}}$ is invariant.

3. Finally, a few exercises concerning the plane-wave solutions $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,x)$ of the Dirac equation.

(a) Show that

$$\sum_{s=1,2} u_a(p,s)\bar{u}_b(p,s) \;=\; (\not{p}+m)_{ab} \quad\text{and}\quad \sum_{s=1,2} v_a(p,s)\bar{v}_b(p,s) \;=\; (\not{p}-m)_{ab}\,. \tag{11}$$

(b) Prove the Gordon identity

$$\bar{u}(p',s')\gamma^\mu u(p.s) \;=\; \frac{(p'+p)^\mu}{2m}\,\bar{u}(p's')u(p,s) \;+\; \frac{i(p'-p)_\nu}{m}\,\bar{u}(p's')S^{\mu\nu}u(p,s). \tag{12}$$

Hint: First, use Dirac equations for the u and the $\bar{u}'$ to show that
$2m\bar{u}'\gamma^\mu u = \bar{u}'(\not{p}'\gamma^\mu + \gamma^\mu\,\not{p})u$.

(c) Generalize the Gordon identity to $\bar{u}'\gamma^\mu v$, $\bar{v}'\gamma^\mu u$ and $\bar{v}'\gamma^\mu v$.

(d) [Added on 10/07] Verify that for $\mathbf{p}' = -\mathbf{p}$, $u^\dagger(p,s)v(p',s') = 0$.