

0. First of all, finish problem #3 from the previous homework set #5. It is due by Thursday, October 7.
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1. The first problem of this set concerns finite representations of the continuous Lorentz symmetry  $SO^+(3, 1)$ , or rather its double cover  $\text{Spin}(3, 1) \cong SL(2, \mathbf{C})$ . Let us define

$$\hat{\mathbf{J}}_{\pm} = \frac{1}{2}(\hat{\mathbf{J}} \pm i\hat{\mathbf{K}}). \quad (1)$$

where  $\hat{K}^i = \hat{J}^{0i} = -\hat{J}^{i0}$  are generators of the Lorentz boosts and  $\hat{J}^i$  are generators of the space rotations.

- (a) Show that the  $\hat{\mathbf{J}}_+$  and the  $\hat{\mathbf{J}}_-$  commute with each other and that each satisfies the commutations relations of an angular momentum,  $[\hat{J}_{\pm}^k, \hat{J}_{\pm}^{\ell}] = i\epsilon^{k\ell m} \hat{J}_{\pm}^m$ .

The “angular momentum”  $\hat{\mathbf{J}}_+$  is non-hermitian and hence its finite irreducible representations are non-unitary analytic continuations of the spin- $j$  representations of a hermitian  $\hat{\mathbf{J}}$ . The same is true for the  $\hat{\mathbf{J}}_- = \hat{\mathbf{J}}_+^{\dagger}$ . Thus altogether, the finite irreducible representations of the Lorentz algebra are specified by two integer or half-integer ‘spins’  $j_+$  and  $j_-$ .

The simplest non-trivial representations of the Lorentz algebra are  $(j_+ = \frac{1}{2}, j_- = 0)$  — the left-handed Weyl spinor where  $\hat{\mathbf{J}}$  acts as  $\frac{1}{2}\boldsymbol{\sigma}$  and  $\hat{\mathbf{K}}$  as  $-\frac{i}{2}\boldsymbol{\sigma}$ , and  $(j_+ = 0, j_- = \frac{1}{2})$  — the right-handed Weyl spinor where  $\hat{\mathbf{J}}$  also acts as  $\frac{1}{2}\boldsymbol{\sigma}$  but  $\hat{\mathbf{K}}$  acts as  $+\frac{i}{2}\boldsymbol{\sigma}$ . Together, the two Weyl spinors comprise the Dirac spinor. From the  $SL(2, \mathbf{C})$  point of view, the left-handed Weyl spinor is the doublet representation  $\mathbf{2}$  which defines the  $SL(2, \mathbf{C})$  group while the right-handed Weyl spinor is the conjugate doublet  $\bar{\mathbf{2}}$ . As discussed in class, the Weyl spinors transform according to

$$\psi_{\alpha}^L \mapsto M_{\alpha}^{\beta} \psi_{\beta}^L \quad \text{and} \quad (\sigma_2 \psi^R)_{\dot{\alpha}} \mapsto M_{\dot{\alpha}}^{*\dot{\beta}} (\sigma_2 \psi^R)_{\dot{\beta}} \quad (2)$$

where  $M \equiv M_L$  and  $\sigma_2 M^* \sigma_2 = M_R$ . Note the notations: the un-dotted indices from the beginning of the Greek alphabet for the left-handed spinors, and the dotted indices for the right-handed spinors.

A generic  $(j_+, j_-)$  representation of the Lorentz algebra becomes in the  $SL(2, \mathbf{C})$  terms a tensor  $\Phi_{\alpha_1 \dots \alpha_{(2j_+)}, \dot{\gamma}_1 \dots \dot{\gamma}_{(2j_-)}}$ , totally symmetric in its  $2j_+$  un-dotted indices  $\alpha_1, \dots, \alpha_{(2j_+)}$  and separately totally symmetric in its  $2j_-$  dotted indices  $\dot{\gamma}_1, \dots, \dot{\gamma}_{(2j_-)}$ ; it transforms according to

$$\Phi_{\alpha_1 \dots \alpha_{(2j_+)}, \dot{\gamma}_1 \dots \dot{\gamma}_{(2j_-)}} \mapsto M_{\alpha_1}^{\beta_1} \dots M_{\alpha_{(2j_+)}}^{\beta_{(2j_+)}} M_{\dot{\gamma}_1}^{*\dot{\delta}_1} \dots U_{\dot{\gamma}_{(2j_-)}}^{*\dot{\delta}_{(2j_-)}} \Phi_{\beta_1 \dots \beta_{(2j_+)}, \dot{\delta}_1 \dots \dot{\delta}_{(2j_-)}}. \quad (3)$$

The vector representation of the Lorentz group has  $j_+ = j_- = \frac{1}{2}$ . To cast the action of the Lorentz group in  $SL(2, \mathbf{C})$  terms (3), we define a  $2 \times 2$  matrix

$$X_\mu \sigma^\mu \equiv X_0 - \mathbf{X} \cdot \boldsymbol{\sigma} \quad (4)$$

where  $\sigma^0$  is a unit  $2 \times 2$  matrix while  $\sigma^1, \sigma^2$  and  $\sigma^3$  are the Pauli matrices. In  $SL(2, \mathbf{C})$  terms, this matrix has one dotted and one un-dotted index,

$$X_{\alpha\dot{\gamma}} = X_\mu \sigma_{\alpha\dot{\gamma}}^\mu = X_0 \delta_{\alpha\dot{\gamma}} - \mathbf{X} \cdot \boldsymbol{\sigma}_{\alpha\dot{\gamma}}, \quad (5)$$

thus it transforms under the  $SL(2, \mathbf{C})$  as a  $(\frac{1}{2}, \frac{1}{2})$  bi-spinor,

$$X'_{\alpha\dot{\gamma}} = M_\alpha^\beta M_{\dot{\gamma}}^{*\dot{\delta}} X_{\beta\dot{\delta}}, \quad (6)$$

or in matrix form

$$X'_\mu \sigma^\mu = M(X_\mu \sigma^\mu) M^\dagger. \quad (7)$$

(b) Show that for any  $SL(2, \mathbf{C})$  matrix  $M$ , eq. (7) defines an orthochronous Lorentz transform  $X'_\mu = L_\mu^\nu(M) X_\nu$ . (Hint: prove and use  $\det(X_\mu \sigma^\mu) = X^2 \equiv X_\mu X^\mu$ ).

\* For extra challenge, show that  $L$  is proper, *i.e.*  $\det(L) = +1$ .

(c) Verify the group law,  $L(M_2 M_1) = L(M_2) L(M_1)$ .

(d) Verify explicitly that for  $M = \exp(-\frac{i}{2} \theta \mathbf{n} \cdot \boldsymbol{\sigma})$ ,  $L(M)$  is a rotation by angle  $\theta$  around axis  $\mathbf{n}$ , while for  $M = \exp(-\frac{1}{2} r \mathbf{n} \cdot \boldsymbol{\sigma})$ ,  $L(M)$  is a boost of rapidity  $r$  ( $\beta = \tanh r$ ,  $\gamma = \cosh r$ ) in the direction  $\mathbf{n}$ .

In general, any  $(j_+, j_-)$  multiplet of the  $SL(2, \mathbf{C})$  with integer net spin  $j_+ + j_-$  is equivalent to some kind of a Lorentz tensor. (Here, we include the scalar and the vector among the tensors.) For example, the  $(1, 1)$  multiplet is equivalent to a symmetric, traceless 2-index tensor  $T^{\mu\nu} = T^{\nu\mu}$ ,  $T^\mu_\mu = 0$ . For  $j_+ \neq j_-$  the representation is complex, but one can make a real tensor by combining two multiplets with opposite  $j_+$  and  $j_-$ , for example the  $(1, 0)$  and  $(0, 1)$  multiplets are together equivalent to an antisymmetric 2-index tensor  $F^{\mu\nu} = -F^{\nu\mu}$ .

(e) Verify the above examples.

Hint: For any angular momentum  $(j = \frac{1}{2}) \otimes (j = \frac{1}{2}) = (j = 1) \oplus (j = 0)$ .

The  $SL(2, \mathbf{C})$  multiplets with half-integer  $j_+ + j_-$  are equivalent to Lorentz spinors or spin-tensors which carry one Weyl index as well as 0, 1 or more 4-vector indices and transform according to

$$\psi_\alpha^{\mu, \dots, \nu} \mapsto M_\alpha^\beta(L) L^\mu_\kappa \cdots L^\nu_\lambda \psi_\beta^{\kappa, \dots, \lambda} \quad \text{or} \quad \psi_{\dot{\alpha}}^{\mu, \dots, \nu} \mapsto M_{\dot{\alpha}}^{*\dot{\beta}}(L) L^\mu_\kappa \cdots L^\nu_\lambda \psi_{\dot{\beta}}^{\kappa, \dots, \lambda}. \quad (8)$$

(f) Show that the  $(1, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$  multiplets are together equivalent to the Rarita-Schwinger spin-vector  $\Psi_a^\mu$  which has one Dirac index  $a$  and one 4-vector index  $\mu$  and satisfies a Lorentz-covariant constraint  $\gamma_\mu \Psi^\mu = 0$ .

2. Now consider an im-proper Lorentz symmetry, namely the *parity* symmetry AKA reflection of space

$$P^\mu_\nu = \begin{cases} +1 & \text{for } \mu = \nu = 0, \\ -1 & \text{for } \mu = \nu = 1, 2, 3, \\ 0 & \text{for } \mu \neq \nu, \end{cases} \quad \text{thus} \quad P(t, \mathbf{x}) = (+t, -\mathbf{x}); \quad (9)$$

note  $P^2 = 1$ .

In the Fock space, parity is represented by a unitary operator  $\hat{\mathcal{P}}$ ; by the group law,  $\hat{\mathcal{P}}^2 = 1$  and hence  $\hat{\mathcal{P}}^\dagger = \hat{\mathcal{P}}^{-1} = \hat{\mathcal{P}}$ .

(a) Use group law to show that  $\hat{\mathcal{P}}$  commutes with the angular momenta  $\hat{J}^i$  but anti-commutes with the boost generators  $\hat{K}^i$ , then use these commutation relations to show that  $\hat{\mathcal{P}}$  acting on the quantum fields must interchange the  $j_+$  and  $j_-$  quantum

numbers of the field components. For example, parity must turn left-handed Weyl spinors into right-handed Weyl spinors and vice versa, thus  $\hat{\mathcal{P}}(\frac{1}{2}, 0)\hat{\mathcal{P}} = (0, \frac{1}{2})$  and  $\hat{\mathcal{P}}(0, \frac{1}{2})\hat{\mathcal{P}} = (\frac{1}{2}, 0)$ .

(b) A Dirac spinor field transforms under parity according to

$$\hat{\mathcal{P}} \hat{\Psi}(\mathbf{x}, t) \hat{\mathcal{P}} \equiv \hat{\Psi}'(\mathbf{x}, t) = \pm \gamma^0 \hat{\Psi}(-\mathbf{x}, t) \quad (10)$$

where the overall  $\pm$  sign is the *intrinsic parity* of a particular Dirac field.

Verify that the Dirac equation is covariant under this transformation and that the Dirac action  $\int d^4x \mathcal{L}_{\text{Dirac}}$  is invariant.

3. Finally, a few exercises concerning the plane-wave solutions  $e^{-ipx}u(p, s)$  and  $e^{+ipx}v(p, s)$  of the Dirac equation.

(a) Show that

$$\sum_{s=1,2} u_a(p, s) \bar{u}_b(p, s) = (\not{p} + m)_{ab} \quad \text{and} \quad \sum_{s=1,2} v_a(p, s) \bar{v}_b(p, s) = (\not{p} - m)_{ab}. \quad (11)$$

(b) Prove the Gordon identity

$$\bar{u}(p', s') \gamma^\mu u(p, s) = \frac{(p' + p)^\mu}{2m} \bar{u}(p', s') u(p, s) + \frac{i(p' - p)_\nu}{m} \bar{u}(p', s') S^{\mu\nu} u(p, s). \quad (12)$$

Hint: First, use Dirac equations for the  $u$  and the  $\bar{u}'$  to show that

$$2m \bar{u}' \gamma^\mu u = \bar{u}' (\not{p}' \gamma^\mu + \gamma^\mu \not{p}) u.$$

(c) Generalize the Gordon identity to  $\bar{u}' \gamma^\mu v$ ,  $\bar{v}' \gamma^\mu u$  and  $\bar{v}' \gamma^\mu v$ .

(d) [Added on 10/07] Verify that for  $\mathbf{p}' = -\mathbf{p}$ ,  $u^\dagger(p, s)v(p', s') = 0$ .