PHY–396 L. Problem set #13. Due January 27, 2005.

1. First, a reading assignment: *Quantum Mechanics and Path Integrals* by Feynman & Hibbs. Read all you can about care and use of Path Integrals.

2. Second, a generalized reading assignment: Refresh your memories of Statistical Mechanics, especially partition functions and related quantities. Use any graduate-level SM book you like.

3. Finally, an actual exercise. Consider a particle on a 1D circle of radius R, or equivalently a 1D particle in a box of length $L = 2\pi R$ with periodic boundary conditions where moving past the $X = L$ points brings you back to $x = 0$. In other words, the particle's position $x(t)$ is defined modulo L.

   (a) Consider all possible particle's paths from a fixed point x_0 (modulo L) at time $t = 0$ to a fixed point x' (modulo L) at time $t = T$. Show that the space of such paths is isomorphic to the space of free particle's paths from a fixed x_0 at $t = 0$ to any of the points $x' + nL$ at $t = T$, for all integer $n = 0, \pm1, \pm2, \ldots$. Then use path integral formalism to show that

   $$U_{\text{box}}(x'; x_0) = \sum_{n=-\infty}^{+\infty} U_{\text{free}}(x' + nL; x_0) \tag{1}$$

   where U_{box} and U_{free} are the evolution kernels (between times $t = 0$ and $t = T$) for the particle in a box and for the free particle.

   According to Poisson re-summation formula, if a function $F(n)$ of integer n can be continued to a function $F(\nu)$ of arbitrary real ν, then

   $$\begin{aligned} \sum_{n=-\infty}^{+\infty} F(n) &= \int d\nu\, F(\nu) \times \sum_{n=-\infty}^{+\infty} \delta(\nu - n) \\ &= \sum_{\ell=-\infty}^{+\infty} \int d\nu\, F(\nu) \times e^{2\pi i \ell \nu}. \end{aligned} \tag{2}$$

(b) The free particle (living on an infinite 1D line) has evolution kernel

$$U_{\text{free}}(x';x_0) \;=\; \sqrt{\frac{M}{2\pi i\hbar T}} \times \exp\left(+\frac{iM(x'-x_0)^2}{2\hbar T}\right). \tag{3}$$

Plug this free kernel into eq. (1) and use Poisson formula to sum over n.

(c) Verify that the resulting evolution kernel for the particle in a box agrees with the usual QM formula

$$U_{\text{box}}(x';x_0) \;=\; \sum_p L^{-1/2} e^{ipx'/\hbar} \times e^{-iT(p^2/2M)/\hbar} \times L^{-1/2} e^{-ipx_0/\hbar} \tag{4}$$

where p takes box-quantized values

$$p \;=\; \frac{2\pi\hbar}{L} \times \text{integer}. \tag{5}$$