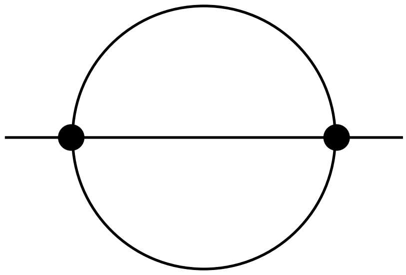
As discussed in class, in the purely scalar field theory, renormalization of the field strength begins at two-loop level. Specifically, the 1PI diagram



(1)

provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence to the $Z - 1$.

Your task is to evaluate this contribution. This is a difficult calculation, so proceed very carefully.

1. First, use Feynman parameters to write the product of 3 propagators as

$$\prod_{j=1}^{3} \frac{i}{q_j^2 - m^2 + i0} = \iiint dx\, dy\, dz\, \delta(x+y+z-1)\, \frac{2i^3}{(\mathcal{D})^3} \tag{2}$$

where

$$\mathcal{D} = xq_1^2 + yq_2^2 + zq_3^2 - m^2 + i0. \tag{3}$$

Then substitute $q_3 = p - q_1 = q_2$ and introduce linear combinations k_1 and k_2 of the q_1, q_2, and p such that $d^4k_1\, d^4k_2 = d^4q_1\, d^4q_2$ and

$$\mathcal{D} = \alpha k_1^2 + \beta k_2^2 + \gamma p^2 - m^2 + i0 \tag{4}$$

for some (x, y, z)–dependent coefficients α, β, γ.

Warning: Do not set $p^2 = m^2$ at this stage.

2. Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1\, d^4k_2\, \frac{1}{\mathcal{D}^4}. \tag{5}$$

Note that although this integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

3. To evaluate the momentum integral (5), first Wick-rotate both momenta k_1 and k_2 to the Euclidean momentum space, and then use dimensional regularization.

Here are some formulæ you will find useful at this stage:

$$\frac{6}{A^4} = \int_0^\infty dt\, t^3\, e^{-At}, \tag{6}$$

$$\int \frac{d^D k}{(2\pi)^D}\, e^{-ctk^2} = \left(4\pi ct\right)^{-D/2}, \tag{7}$$

$$\Gamma(2\epsilon)X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \tfrac{1}{2}\log X. \tag{8}$$

4. At this stage, you should arrive at a formula which looks like

$$\frac{d\Sigma}{dp^2} = \iiint dxdydz\, \delta(x+y+z-1)\, F(x,y,z) \times \times \left\{ \frac{1}{\epsilon} + \log\frac{\mu^2}{m^2} + \text{const} + \log G(x,y,z;p^2/m^2) \right\} \tag{9}$$

for some rational functions F, G of the Feynman parameters (and in case of G, also of p^2/m^2). Our goal is the field strength renormalization factor

$$Z = \left[1 - \frac{d\Sigma}{dp^2}\right]^{-1} \tag{10}$$

where the derivative is evaluated at $p^2 = M_{\text{ph}}^2$ — the physical mass^2 of the scalar particle. To the leading approximation, we may let $M_{\text{ph}}^2 \approx m^2$, — and that should simplify the $G(x,y,z)$ function in eq. (9) just a bit.

Nevertheless, the Feynman parameter integral (9) remains a frightful mess. Do not try to evaluate it by hand — it would take way too much time. Instead, you should use Mathematica (or equivalent software). To help it along, replace the (x, y, z) variables with (w, ξ) according to $x = \xi w$, $y = (1 - \xi)w$, $z = 1 - w$, then integrate over the w variable first and over the ξ second. Here is a couple of integrals I did this way you might find useful:

$$\begin{aligned}
\iiint dxdydz\,\delta(x+y+z-1)\times\frac{xyz}{(xy+xz+yz)^3} &= \frac{1}{2}\,,\\
\iiint dxdydz\,\delta(x+y+z-1)\times\frac{xyz}{(xy+xz+yz)^3}\,\log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} &= -\frac{3}{4}\,.
\end{aligned} \tag{11}$$