1. This homework is about renormalization of the *scalar QED* — the theory of a *charged* scalar field $\Phi(x)$ coupled to the EM field $A_\mu(x)$.

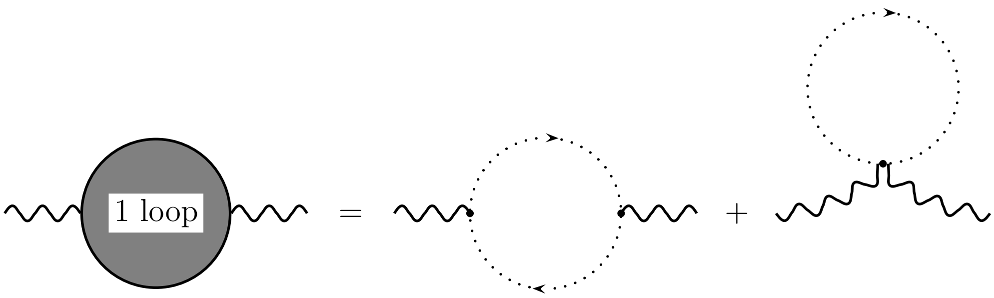
   (a) Write down all counterterms needed to cancel all the divergences of the scalar QED. Start with the superficial divergences, and then use Ward identities to narrow down the list of actually divergent amplitudes.

   (b) Show that all the requisite counterterms are contained in a renormalizable bare Lagrangian.

   (c) Write down renormalization conditions for the finite parts of the counterterms.

   (d) In ordinary QED, gauge invariance of the counterterms requires $\delta_1 = \delta_2$. In the scalar QED, gauge invariance requires two identities of this kind. Write them down, and prove them using Ward identities of the un-amputated amplitudes, *cf.* homework set #15.

2. And now consider the electric charge renormalization in the scalar QED. At the one-loop level, there are two Feynman diagrams contributing to the 1PI two-photon amplitude, namely

$-i\Sigma^{\mu\nu}_{1\,\text{loop}}$ = 

   (a) Evaluate the two diagrams using dimensional regularization and verify that the net amplitude has form

$$\Sigma^{\mu\nu}_{1\,\text{loop}}(k) \;=\; \left(k^2 g^{\mu\nu} \,-\, k^\mu k^\nu\right) \times \Pi_{1\,\text{loop}}(k^2) \tag{1}$$

(b) Calculate the $\Pi(k^2)$ due to the above diagrams, determine the δ_3 counterterm (at the one-loop level), and write down the *net* $\Pi(k^2)$ as a function of k^2.

(c) Finally, consider the effective coupling $\alpha_{\text{eff}}(k^2)$ of the scalar QED at high momenta. Show that at the one-loop level,

$$\frac{1}{\alpha_{\text{eff}}(k^2)} \approx \frac{1}{\alpha(0)} - \frac{1}{12\pi}\left(\log\frac{-k^2}{m^2} - \frac{8}{3}\right). \tag{2}$$