

1. First, a couple of reading assignments:
 - (a) Read about the Wilsonian renormalization: §12.1 of Peskin & Schroeder book, and §12.4 of Weinberg's book (vol. 1).
 - (b) Refresh your knowledge of basic group theory, esp. Lie groups, Lie Algebras, and their representations.
2. In class, we discussed field multiplets $\Psi_i(x)$ which transform as (complex) vectors under the $SU(N)$ symmetry,

$$\Psi'(x) = U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) = \sum_j U_i^j(x)\Psi_j(x), \quad i, j = 1, 2, \dots, N \quad (1)$$

where $U(x)$ is an x -dependent unitary $N \times N$ matrix, $\det U(x) \equiv 1$. Now consider the *adjoint multiplet* $\Phi_i^j(x)$ of fields: for each x , it comprises a traceless hermitian $N \times N$ matrix $\Phi(x)$ which transforms according to

$$\Phi'(x) = U(x)\Phi(x)U^\dagger(x), \quad i.e. \quad \Phi_i^j(x) = \sum_{k,\ell} U_i^k(x)\Phi_k^\ell(x)U_\ell^{\dagger j}(x). \quad (2)$$

Note that this transformation law preserves the $\Phi^\dagger = \Phi$ and $\text{tr}(\Phi) = 0$ conditions.

The covariant derivative acts on the adjoint multiplet according to

$$D_\mu\Phi(x) = \partial_\mu(x) + i[A_\mu(x), \Phi(x)] \equiv \partial_\mu(x) + iA_\mu(x)\Phi(x) - i\Phi(x)A_\mu(x) \quad (3)$$

- (a) Verify that this derivative is indeed covariant and $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like $\Phi(x)$ itself.
- (b) Show that $[D_\mu, D_\nu]\Phi(x) = i[F_{\mu\nu}(x), \Phi(x)]$.

The non-abelian tension field $F_{\mu\nu}(x)$ itself transforms according to the adjoint representation of the local symmetry, $F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x)$. Hence, the covariant derivative acts on the tension field according to $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + i[A_\lambda, F_{\mu\nu}]$.

- (c) Verify the non-abelian Bianchi identity $D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0$.
- (d) Show that for an infinitesimal variation of the non-abelian gauge field $A_\nu(x) \rightarrow A_\nu(x) + \delta A_\nu(x)$, the tension varies according to $\delta F_{\mu\nu}(x) = D_\mu \delta A_\nu(x) - D_\nu \delta A_\mu(x)$.
- (e) Finally, consider the classical non-abelian gauge theory comprising the gauge fields $A_i^{\mu j}(x)$ and a vector multiplet of Dirac fields $\Psi_i(x)$. In matrix notations, the Lagrangian is

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi. \quad (4)$$

Write down the classical equations of motion for this theory.