1. First, a couple of reading assignments:

   (a) Read about the Wilsonian renormalization: §12.1 of Peskin & Schroeder book, and §12.4 of Weinberg's book (vol. 1).

   (b) Refresh your knowledge of basic group theory, esp. Lie groups, Lie Algebras, and their representations.

2. In class, we discussed field multiplets $\Psi_i(x)$ which transform as (complex) vectors under the $SU(N)$ symmetry,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;=\sum_j U_i^{\;j}(x)\Psi_j(x), \quad i,j=1,2,\ldots,N \tag{1}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x)\equiv 1$. Now consider the *adjoint multiplet* $\Phi_i^{\;j}(x)$ of fields: for each x, it comprises a traceless hermitian $N\times N$ matrix $\Phi(x)$ which transforms according to

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x), \quad i.e. \quad \Phi_i^{\prime j}(x) \;=\sum_{k,\ell} U_i^{\;k}(x)\,\Phi_k^{\;\ell}(x)\,U_\ell^{\dagger j}(x). \tag{2}$$

Note that this transformation law preserves the $\Phi^\dagger=\Phi$ and $\mathrm{tr}(\Phi)=0$ conditions.

The covariant derivative acts on the adjoint multiplet according to

$$D_\mu\Phi(x) \;=\; \partial_\mu(x) \;+\; i[A_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu(x) \;+\; iA_\mu(x)\Phi(x) \;-\; i\Phi(x)A_\mu(x) \tag{3}$$

   (a) Verify that this derivative is indeed covariant and $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like $\Phi(x)$ itself.

   (b) Show that $[D_\mu,D_\nu]\Phi(x)=i[F_{\mu\nu}(x),\Phi(x)]$.

The non-abelian tension field $F_{\mu\nu}(x)$ itself transforms according to the adjoint representation of the local symmetry, $F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x)$. Hence, the covariant derivative acts on the tension field according to $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + i[A_\lambda, F_{\mu\nu}]$.

(c) Verify the non-abelian Bianchi identity $D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0$.

(d) Show that for an infinitesimal variation of the non-abelian gauge field $A_\nu(x) \to A_\nu(x) + \delta A_\nu(x)$, the tension varies according to $\delta F_{\mu\nu}(x) = D_\mu \delta A_\nu(x) - D_\nu \delta A_\mu(x)$.

(e) Finally, consider the classical non-abelian gauge theory comprising the gauge fields $A_i^{\mu j}(x)$ and a vector multiplet of Dirac fields $\Psi_i(x)$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \;+\; \overline{\Psi}(i\gamma^\mu D_\mu \;-\; m)\Psi. \tag{4}$$

Write down the classical equations of motion for this theory.