

1. First, a bit of group theory. Consider a generic simple non-abelian Lie group G and its generators T^a . The (quadratic) Casimir operator $C_2 = \sum_a T^a T^a$ commutes with all the generators and hence for any irreducible representation (r) of the group, C_2 restricted to (r) is simply a unit matrix times a number $C(r)$. In other words, if $T_{(r)}^a$ is a matrix of the generator T^a in the representation (r), then $\sum_a T_{(r)}^a T_{(r)}^a = C(r) \times \mathbf{1}$. For example, for the isospin group $SU(2)$, the irreps are characterized by the isospin I and $C(I) = I(I + 1)$.

(a) By symmetry, for any complete representation (r) of the group,

$$\text{tr}_{(r)}(T^a T^b) \equiv \text{tr} \left(T_{(r)}^a T_{(r)}^b \right) = R(r) \delta^{ab} \quad (1)$$

for some coefficient $R(r)$. Show that for any irreducible representation,

$$\frac{R(r)}{C(r)} = \frac{\dim(r)}{\dim(G)}. \quad (2)$$

In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I + 1)(2I + 1)$.

(b) Suppose the first three generators of G generate an $SU(2)$ subgroup. Show that if a representation (r) of G decomposes into several $SU(2)$ multiplets of isospins I_1, I_2, \dots, I_n , then

$$R(r) = \sum_{i=1}^n \frac{1}{3} I_i (I_i + 1) (2I_i + 1). \quad (3)$$

(c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting on the first two components of a complex N -vector. The fundamental representation (N) of the $SU(N)$ decomposes into one doublet and $(N - 2)$ singlets of the $SU(2)$ subgroup, hence

$$R(N) = \frac{1}{2} \quad \text{and} \quad C(N) = \frac{N^2 - 1}{2N}. \quad (4)$$

Show that the adjoint representation of the $SU(N)$ decomposes into one $SU(2)$

triplet, $2(N - 2)$ doublets and $(N - 2)^2$ singlets and hence

$$R(\text{adj}) = C(\text{adj}) \equiv C(G) = N. \quad (5)$$

Hint: $(N) \times (\bar{N}) = (\text{adj}) + (1)$.

(d) The symmetric and the anti-symmetric 2-index tensors form irreducible representations of the $SU(N)$ group. Find out the decomposition of these irreps under an $SU(2) \subset SU(N)$ and calculate their respective R factors.

2. And now let's apply group theory to a physical process of quark-antiquark pair production in *Quantum Chromodynamics* (QCD). Specifically, let us focus on the $u\bar{u} \rightarrow d\bar{d}$ process so there is only one tree-level diagram contributing to this process. Draw this diagram and calculate the amplitude, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles and calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the $u\bar{u} \rightarrow d\bar{d}$ pair production in QCD is very similar to the $e^-e^+ \rightarrow \mu^-\mu^+$ pair production in QED, so the only new aspect of this problem is summing over the colors.

3. Next, consider a scalar analogue of QCD or more generally a theory of Yang-Mills fields A_μ^a and complex scalars Φ_i in some representation (r) of the gauge group G .

(a) Write down the Lagrangian and the Feynman rules of this theory.

Next, consider the annihilation process $\Phi + \Phi^* \rightarrow 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

(b) Draw the diagrams and write down the tree-level annihilation amplitude.

As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving a longitudinally polarized gauge boson vanish, provided all other gauge bosons are transversely polarized.* In other words,

$$\begin{aligned} \mathcal{M} &\equiv e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n} \mathcal{M}_{\mu_1 \mu_2 \cdots \mu_n}(\text{momenta}) = 0 \\ \text{when } e_1^\mu &\propto k_1^\mu \quad \text{but} \quad e_2^\nu k_{2\nu} = \cdots = e_n^\nu k_{n\nu} = 0. \end{aligned}$$

(c) Verify this identity for the scalar annihilation amplitude.

4. To convert the annihilation amplitude into a cross-section we need to sum / average over the colors of all the particles. As a first step in this direction, it's convenient to write the amplitude as

$$\mathcal{M}(j + i \rightarrow a + b) = F \times \{T^a, T^b\}_j^i + iG \times [T^a, T^b]_j^i \quad (6)$$

where j is the 'color' index of the scalar particle belonging to some representation (r) of the gauge group G , i is the color index of the scalar anti-particle belonging to the conjugate representation (\bar{r}), and a and b are the color indices of the gauge bosons belonging to the adjoint representation.

- (a) Show that the annihilation amplitude indeed has form (6) and write down the coefficients F and G as explicit functions of the particles momenta and polarizations.
- (b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{C(r)}{\dim(r)} \times (4C(r) \times |F|^2 + C(\text{adj}) \times (|G|^2 - |F|^2)). \quad (7)$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 = \frac{N^2 - 1}{2N^2} \left(\frac{N^2 - 2}{N} |F|^2 + N|G|^2 \right). \quad (8)$$

- (c) Evaluate F and G in the center of mass frame. In this frame, the vector particles' polarizations $e_{1,2}^\mu = (0, \mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors momenta $\pm \mathbf{k}$. For simplicity, use planar rather than circular polarizations.
- (d) Finally, calculate the (polarized, partial) cross-section for the annihilation process.