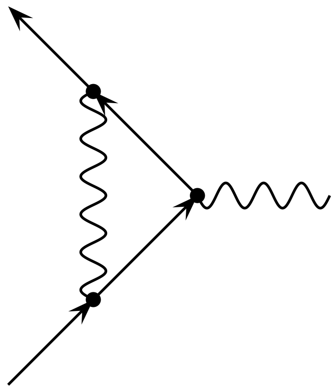
QED Vertex Correction: Working through the Algebra

At the one-loop level of QED, the 1PI vertex correction comes from a single Feynman diagram



(1)

thus

$$ie\Gamma^{\mu}_{1\,\mathrm{loop}}(p',p) = \int\limits_{\mathrm{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{-ig^{\nu\lambda}}{k^2+i0} \times ie\gamma_{\nu} \times \frac{i}{\not p' + \not k - m + i0} \times ie\gamma^{\mu} \times \frac{i}{\not p + \not k - m + i0} \times ie\gamma_{\lambda}$$

$$= e^3 \int\limits_{\mathrm{reg}} \frac{d^4k}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu}}{\mathcal{D}} \qquad (2)$$

where

$$\mathcal{N}^{\mu} = \gamma^{\nu}(\not k + \not p' + m)\gamma^{\mu}(\not k + \not p + m)\gamma_{\nu} \qquad (3)$$

and

$$\mathcal{D} = [k^2 + i0] \times [(p+k)^2 - m^2 + i0] \times [(p'+k)^2 - m^2 + i0]. \qquad (4)$$

Using Feynman parameter trick, we re-write the denominator as

$$\frac{1}{\mathcal{D}} = \iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2}{\Big[x((p+k)^2-m^2)\ +\ y((p'+k)^2-m^2)\ +\ z(k^2)\ +\ i0\Big]^3} \qquad (5)$$

and then expand

$$x((p+k)^2-m^2)\;+\;y((p'+k)^2-m^2)\;+\;z(k^2)\;=\;\ell^2\;-\;\Delta \tag{6}$$

where

$$\ell\;=\;k\;+\;xp\;+\;yp' \tag{7}$$

and

$$\Delta\;=\;(xp+yp')^2\;+\;x(m^2-p^2)\;+\;y(m^2-p'^2)\;=\;(1-z)^2m^2\;-\;xyq^2\quad\langle\!\langle\text{on shell}\rangle\!\rangle \tag{8}$$

Therefore

$$\Gamma^{\mu}_{1\,\text{loop}}(p',p)\;=\;-2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int_{\text{reg}}\frac{d^4\ell}{(2\pi)^4}\,\frac{\mathcal{N}^{\mu}}{\big[\ell^2-\Delta+i0\big]^3}\,, \tag{9}$$

and now we need to simplify the numerator (3) *in the context of this monstrous integral.*

The first step is obvious: Let us get rid of the γ^ν and γ_ν factors using the γ matrix algebra, *eg,* $\gamma^\nu\not{a}\gamma_\nu=-2\not{a}$, *etc.*. However, in order to allow for the dimensional regularization, we need to re-work the algebra for an arbitrary spacetime dimension D where $\gamma^\nu\gamma_\nu=D\neq 4$. Consequently,

$$\begin{aligned}\gamma^\nu\not{a}\gamma_\nu\;&=\;-2\not{a}\;+\;(4-D)\not{a},\\ \gamma^\nu\not{a}\not{b}\gamma_\nu\;&=\;4(ab)\;-\;(4-D)\not{a}\not{b},\\ \gamma^\nu\not{a}\not{b}\not{c}\gamma_\nu\;&=\;-2\not{c}\not{b}\not{a}\;+\;(4-D)\not{a}\not{b}\not{c},\end{aligned} \tag{10}$$

and therefore

$$\mathcal{N}^\mu\;=\;-2m^2\gamma^\mu\;+\;4m(p'+p+2k)^\mu\;-\;2(\not{p}+\not{k})\gamma^\mu(\not{p}'+\not{k})\;+\;(4-D)(\not{p}'+\not{k}-m)\gamma^\mu(\not{p}+\not{k}-m). \tag{11}$$

Next, we re-express the right hand side here in terms of the Feynman's loop momentum ℓ rather than k using eq. (7). Expanding the result in powers of ℓ, we get quadratic, linear and

ℓ–independent terms, but the linear terms fo not contribute to the $\int d^D\ell$ integral because they are odd with respect to $\ell \to -\ell$ while everything else in that integral is even. Consequently, in the context of eq. (9), we may neglect the linear terms, thus

$$\begin{aligned}
\mathcal{N}^\mu \;&\cong\; -2m^2\gamma^\mu \;+\; 4m(p+p'-2xp-2yp')^\mu \\
&\quad -\; 2\,\not\ell\gamma^\mu\not\ell \;-\; 2(\not p - x\not p - y\not p')\,\gamma^\mu\,(\not p' - x\not p - y\not p') \\
&\quad +\; (4-D)\,\not\ell\gamma^\mu\not\ell \;+\; (4-D)(\not p' - y\not p' - x\not p - m)\,\gamma^\mu\,(\not p - x\not p - y\not p' - m) \\
&=\; -2m^2\gamma^\mu \;+\; 4mz(p'+p)^\mu \;+\; 4m(x-y)q^\mu \;-\; (D-2)\,\not\ell\gamma^\mu\not\ell \\
&\quad -\; 2(z\not p' + (x-1)\not q)\,\gamma^\mu\,(z\not p + (1-y)\not q) \\
&\quad +\; (4-D)(z\not p' + x\not q - m)\,\gamma^\mu\,(z\not p - y\not q - m)
\end{aligned} \tag{12}$$

where the second equality here follows from $p'-p=q$ and $x+y+z=1$.

Now, let make use of the external fermions being on-shell. This means more than just $p^2 = p'^2 = m^2$: Effectively, we sandwich the vertex $ie\Gamma^\mu$ between Dirac spinors $\bar u(p')$ on the left and $u(p)$ on the right. The two spinors satisfy the appropriate Dirac equations, and hence any term in Γ^μ something$\times\not p$ is equivalent to same thing $\times\, m$ because $\not p u(p) = mu(p)$, and likewise any term of the form $\not p' \times$ something is equivalent to $m\times$ same thing because $\bar u(p')\not p' = \bar u(p')m$. Consequently, the terms on the last two lines of eq. (12) are equivalent to

$$\begin{aligned}
(z\not p' + (x-1)\not q)\,\gamma^\mu\,(z\not p + (1-y)\not q) \;&\cong\; (zm + (x-1)\not q)\,\gamma^\mu\,(zm + (1-y)\not q) \\
(z\not p' + x\not q - m)\,\gamma^\mu\,(z\not p - y\not q - m) \;&\cong\; ((z-1)m + x\not q)\,\gamma^\mu\,((z-1)m - y\not q).
\end{aligned} \tag{13}$$

Let us combine these two expressions with respective coefficients -2 and $4-D$ (*cf.* eq. (12)) and group similar terms together. Making use of

$$\not q\gamma^\mu \;=\; q^\mu \;+\; i\sigma^{\mu\nu}q_\nu \quad\text{and}\quad \gamma^\mu\not q \;=\; q^\mu \;-\; i\sigma^{\mu\nu}q_\nu\,, \tag{14}$$

we obtain

$$\begin{aligned}
&m^2\gamma^\mu\times\Big(-2z^2 + (4-D)(1-z)^2\Big) \\
&+\; \not q\gamma^\mu\not q\times\Big(2(1-x)(1-y) - (4-D)xy\Big) \\
&+\; mq^\mu\times(x-y)\Big(-2z - (4-D)(1-z)\Big) \\
&+\; im\sigma^{\mu\nu}q_\nu\times\Big(2z(2-x-y) - (4-D)(1-z)(x+y)\Big),
\end{aligned} \tag{15}$$

and hence

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;+\; 4mz(p'+p)^\mu \\ & +\; m^2\gamma^\mu\times\Big(-2-2z^2+(4-D)(1-z)^2\Big) \\ & +\; \not{q}\gamma^\mu\not{q}\times\Big(2(1-x)(1-y)-(4-D)xy\Big) \\ & +\; mq^\mu\times(x-y)\Big(4-2z-(4-D)(1-z)\Big) \\ & +\; im\sigma^{\mu\nu}q_\nu\times\Big(2z(2-x-y)-(4-D)(1-z)(x+y)\Big).\end{aligned} \tag{16}$$

Furthermore, in the context of the Dirac sandwich $\bar{u}(p')\Gamma^\mu u(p)$ we have

$$\not{q}\gamma^\mu\not{q} \;=\; 2q^\mu\,\not{q} \;-\; q^2\gamma^\mu \;\cong\; -q^2\gamma^\mu \tag{17}$$

because $\bar{u}(p')\,\not{q}u(p)=0$, and also the Gordon identity

$$(p'+p)^\mu \;\cong\; 2m\gamma^\mu \;-\; i\sigma^{\mu\nu}q_\mu\,. \tag{18}$$

Therefore, re-grouping terms and making use of $x+y+z=1$, we obtain

$$\begin{aligned}\mathcal{N}^\mu \;\cong\; & -(D-2)\,\not{\ell}\gamma^\mu\not{\ell} \;-\; m^2\gamma^\mu\times\Big(2(1-4z+z^2)-(4-D)(1-z)^2\Big) \\ & -\; q^2\gamma^\mu\times\Big(2(z+xy)-(4-D)xy\Big) \;-\; im\sigma^{\mu\nu}q_\nu\times(1-z)\Big(2z+(4-D)(1-z)\Big) \\ & +\; mq^\mu\times(x-y)\Big(4-2z-(4-D)(1-z)\Big).\end{aligned} \tag{19}$$

To further simplify this expression, let us go back to the symmetries of the integral (9). The integral over the Feynman parameters, the integral $\int d^D\ell$, and the denominator $[l^2-\Delta]^3$ are all invariant under the parameter exchange $x\leftrightarrow y$. In eq. (19) for the numerator, the first two lines are invariant under this symmetry, but the last line changes sign. Consequently, only the first two lines contribute to the integral (9) while the third line integrates to zero and may be disregarded.

Finally, thanks to the Lorentz invariance of the $\int d^D\ell$ integral,

$$\ell_\lambda \ell_\nu \;\cong\; g_{\lambda\nu} \times \frac{\ell^2}{D}\,, \tag{20}$$

and hence

$$\not\ell \gamma^\mu \not\ell \;=\; \gamma^\lambda\gamma^\mu\gamma^\nu \times \ell_\lambda\ell_\nu \;\cong\; \gamma^\lambda\gamma^\mu\gamma^\nu \times g_{\lambda\nu}\,\frac{\ell^2}{D} \;=\; -(D-2)\gamma^\mu \times \frac{\ell^2}{D}\,. \tag{21}$$

Plugging this formula into eq. (19) and grouping terms according to their γ–matrix structure, we arrive at

$$\mathcal{N}^\mu \;=\; \mathcal{N}_1 \times \gamma^\mu \;-\; \mathcal{N}_2 \times \frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{22}$$

where

$$\begin{aligned} \mathcal{N}_1 \;&\cong\; \frac{(D-2)^2}{D}\,\ell^2 \;-\; 2(1-4z+z^2)\,m^2 \;-\; 2(z+xy)q^2 \\ &\qquad\qquad\quad +\; (4-D)((1-z)^2m^2+xyq^2), \end{aligned} \tag{23}$$

$$\mathcal{N}_2 \;\cong\; 4z(1-z)m^2 \;+\; 2(4-D)(1-z)^2m^2. \tag{24}$$

Note that splitting the numerator according to eq. (22) is particularly convenient for calculating the electron's form factors:

$$\Gamma^\mu_{1\,\text{loop}} \;=\; F_1^{1\,\text{loop}}(q^2)\times\gamma^\mu \;+\; F_2^{1\,\text{loop}}(q^2)\times\frac{i\sigma^{\mu\nu}q_\nu}{2m}\,, \tag{25}$$

$$F_1^{1\,\text{loop}}(q^2) \;=\; -2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_1}{\left[\ell^2-\Delta+i0\right]^3}\,, \tag{26}$$

$$F_2^{1\,\text{loop}}(q^2) \;=\; +2ie^2\iiint_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int\frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}_2}{\left[\ell^2-\Delta+i0\right]^3}\,. \tag{27}$$