Problem **3**(a):

First, let us calculate the commutator $[\gamma^\kappa\gamma^\lambda,\gamma^\nu]$. According to eq. (18), any product of 2 Dirac matrices can be re-ordered as

$$\gamma^\nu\gamma^\mu \;=\; 2g^{\mu\nu} \;-\; \gamma^\mu\gamma^\nu. \tag{S.1}$$

Applying this formula twice, we have

$$\gamma^\kappa\gamma^\lambda\gamma^\mu \;=\; \gamma^\kappa(2g^{\lambda\mu}-\gamma^\mu\gamma^\lambda) \;=\; 2g^{\lambda\mu}\gamma^\kappa \;-\; (\gamma^\kappa\gamma^\mu)\gamma^\lambda \;=\; 2g^{\lambda\mu}\gamma^\kappa \;-\; 2g^{\kappa\mu}\gamma\lambda \;+\; \gamma^\mu\gamma^\kappa\gamma^\lambda \tag{S.2}$$

and hence

$$[\gamma^\kappa\gamma^\lambda,\gamma^\nu] \;=\; 2g^{\lambda\mu}\gamma^\kappa \;-\; 2g^{\kappa\mu}\gamma^\lambda. \tag{S.3}$$

Next, according to eq. (S.1), $[\gamma^\mu,\gamma^\nu]=2\gamma^\mu\gamma^\nu-2g^{\mu\nu}$ and hence

$$S^{\mu\nu} \;=\; -\tfrac{i}{2}(\gamma^\mu\gamma^\nu \;-\; g^{\mu\nu}). \tag{S.4}$$

Consequently,

$$[S^{\kappa\lambda},\gamma^\mu] \;=\; -\tfrac{i}{2}[\gamma^\kappa\gamma^\lambda,\gamma^\mu] \;=\; -ig^{\lambda\mu}\gamma^\kappa \;+\; ig^{\kappa\mu}\gamma^\lambda \tag{20}$$

or equivalently

$$[\gamma^\mu,S^{\kappa\lambda}] \;=\; ig^{\lambda\mu}\gamma^\kappa \;-\; ig^{\kappa\mu}\gamma^\lambda. \tag{S.5}$$

Finally, by Leibniz rule,

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda,S^{\mu\nu}\right] \;&=\; \gamma^\kappa\left[\gamma^\lambda,S^{\mu\nu}\right] \;+\; \left[\gamma^\kappa,S^{\mu\nu}\right]\gamma^\lambda\\
&=\; \gamma^\kappa\bigl(ig^{\lambda\mu}\gamma^\nu \;-\; ig^{\lambda\nu}\gamma^\mu\bigr) \;+\; \bigl(ig^{\kappa\mu}\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\bigr)\gamma^\lambda\\
&=\; ig^{\lambda\mu}\gamma^\kappa\gamma^\nu \;-\; ig^{\kappa\nu}\gamma^\mu\gamma^\lambda \;-\; ig^{\lambda\nu}\gamma^\kappa\gamma^\mu \;+\; ig^{\kappa\mu}\gamma^\nu\gamma^\lambda\\
&=\; ig^{\lambda\mu}\bigl(\gamma^\kappa\gamma^\nu-g^{\kappa\nu}\bigr) \;-\; ig^{\kappa\nu}\bigl(\gamma^\mu\gamma^\lambda-g^{\lambda\mu}\bigr)\\
&\qquad -\; ig^{\lambda\nu}\bigl(\gamma^\kappa\gamma^\mu-g^{\kappa\mu}\bigr) \;+\; ig^{\kappa\mu}\bigl(\gamma^\nu\gamma^\lambda-g^{\lambda\nu}\bigr)\\
&=\; 2g^{\lambda\mu}S^{\kappa\nu} \;-\; 2g^{\kappa\nu}S^{\mu\lambda} \;-\; 2g^{\lambda\nu}S^{\kappa\mu} \;+\; 2g^{\kappa\mu}S^{\nu\lambda},
\end{aligned} \tag{S.6}$$

and therefore,

$$\begin{aligned}\left[S^{\kappa\lambda},S^{\mu\nu}\right] &= \tfrac{i}{2}\left[\gamma^\kappa\gamma^\lambda,S^{\mu\nu}\right]\\ &= ig^{\lambda\mu}S^{\kappa\nu} - ig^{\kappa\nu}S^{\mu\lambda} - ig^{\lambda\nu}S^{\kappa\mu} + ig^{\kappa\mu}S^{\nu\lambda}\\ &= ig^{\lambda\mu}S^{\kappa\nu} + ig^{\kappa\nu}S^{\lambda\mu} - ig^{\lambda\nu}S^{\kappa\mu} - ig^{\kappa\mu}S^{\lambda\nu}.\end{aligned} \tag{21}$$

$\mathcal{Q.E.D.}$

Problem $\mathbf{3}$(b):

$$\begin{aligned}\left\{\gamma^\rho,\gamma^\lambda\gamma^\mu\gamma^\nu\right\} &= 2g^{\rho\lambda}\,\gamma^\mu\gamma^\nu - 2g^{\rho\mu}\,\gamma^\lambda\gamma^\nu + 2g^{\rho\nu}\,\gamma^\lambda\gamma^\mu,\\ \left[\gamma^\rho,\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu\right] &= 2g^{\rho\kappa}\,\gamma^\lambda\gamma^\mu\gamma^\nu - 2g^{\rho\lambda}\,\gamma^\kappa\gamma^\mu\gamma^\nu + 2g^{\rho\mu}\,\gamma^\kappa\gamma^\lambda\gamma^\nu - 2g^{\rho\nu}\,\gamma^\kappa\gamma^\lambda\gamma^\mu,\\ \left[S^{\rho\sigma},\gamma^\lambda\gamma^\mu\gamma^\nu\right] &= ig^{\sigma\lambda}\,\gamma^\rho\gamma^\mu\gamma^\nu + ig^{\sigma\mu}\,\gamma^\lambda\gamma^\rho\gamma^\nu + ig^{\sigma\nu}\,\gamma^\lambda\gamma^\mu\gamma^\rho\\ &\quad - ig^{\rho\lambda}\,\gamma^\sigma\gamma^\mu\gamma^\nu - ig^{\rho\mu}\,\gamma^\lambda\gamma^\sigma\gamma^\nu - ig^{\rho\nu}\,\gamma^\lambda\gamma^\mu\gamma^\sigma.\end{aligned}$$

The algebra is straightforward.

Problem $\mathbf{3}$(c):

$$\begin{aligned}\gamma^\alpha\gamma_\alpha &= \tfrac12\{\gamma^\alpha,\gamma^\beta\}g_{\alpha\beta} = g^{\alpha\beta}g_{\alpha\beta} = 4;\\ \gamma^\alpha\gamma^\nu\gamma_\alpha &= 2g^{\alpha\nu}\gamma_\alpha - \gamma^\nu\gamma^\alpha\gamma_\alpha = 2\gamma^\nu - \gamma^\nu(4) = -2\gamma^\nu;\\ \gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha &= 2g^{\alpha\mu}\gamma^\nu\gamma_\alpha - \gamma^\mu\gamma^\alpha\gamma^\nu\gamma_\alpha = 2\gamma^\nu\gamma^\mu - \gamma^\mu(-2\gamma^\nu) = 2\{\gamma^\nu,\gamma^\mu\} = 4g^{\mu\nu};\\ \gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha &= 2g^{\alpha\lambda}\gamma^\mu\gamma^\nu\gamma_\alpha - \gamma^\lambda\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 2\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\lambda(4g^{\mu\nu})\\ &= 2\big(\gamma^\mu\gamma^\nu - 2g^{\mu\nu}\big)\gamma^\lambda = -2\gamma^\nu\gamma^\mu\gamma^\lambda.\end{aligned} \tag{S.7}$$

Problem $\mathbf{3}$(d): Let $F=-\frac{i}{2}\Theta_{\alpha\beta}S^{\alpha\beta}$, thus $M=e^F$ and $M^{-1}=e^{-F}$. We shall use the multiple-commutator formula for the $e^{-F}\gamma^\mu e^{+F}$, so we begin by evaluating the single commutator

$$\left[\gamma^\mu,F\right] = -\tfrac{i}{2}\Theta_{\alpha\beta}\left[\gamma^\mu,S^{\alpha\beta}\right] = \tfrac12\Theta_{\alpha\beta}\big(g^{\mu\alpha}\gamma^\beta - g^{\mu\beta}\gamma^\alpha\big) = \Theta_{\alpha\beta}\,g^{\mu\alpha}\gamma^\beta = \Theta^\mu{}_\beta\gamma^\beta. \tag{S.8}$$

The multiple commutators follow immediately from this formula,

$$\begin{aligned} \big[[\gamma^\mu, F], F\big] &= \Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \nu}\gamma^\nu, \\ \big[\big[[\gamma^\mu, F], F\big], F\big] &= \Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \rho}\Theta^\rho_{\ \nu}\gamma^\nu, \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots \end{aligned} \tag{S.9}$$

Therefore, by the multiple commutator formula,

$$\begin{aligned} M^{-1}\gamma^\mu M &= e^{-F}\gamma^\mu e^{+F} \\ &= \gamma^\mu + \big[\gamma^\mu, F\big] + \tfrac{1}{2}\big[[\gamma^\mu, F], F\big] + \tfrac{1}{6}\big[\big[[\gamma^\mu, F], F\big], F\big] + \cdots \\ &= \gamma^\mu + \Theta^\mu_{\ \nu}\gamma^\nu + \tfrac{1}{2}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \nu}\gamma^\nu + \tfrac{1}{6}\Theta^\mu_{\ \lambda}\Theta^\lambda_{\ \rho}\Theta^\rho_{\ \nu}\gamma^\nu + \cdots \\ &= L^\mu_{\ \nu}\gamma^\nu. \end{aligned} \tag{S.10}$$

$\mathcal{Q}.\mathcal{E}.\mathcal{D}.$