Problem **3**(a):

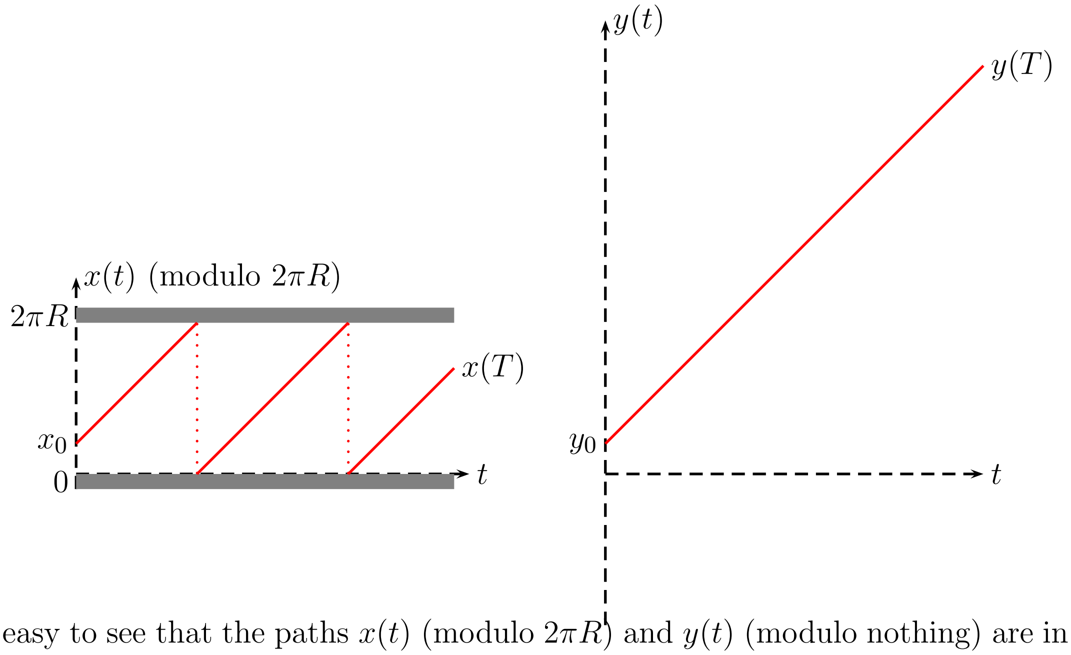
The difference between a circle and a straight line is that on a circle the path of a particle going from point x_0 to point x' does not need to be 'straight' but may wrap around the whole circle one or more times. Indeed, let us compare a particle moving on a circle according to $x(t)$ (modulo $2\pi R$) with a particle moving on an infinite line according to $y(t)$. If the two particles have exactly the same velocities at all times,

$$\frac{dx}{dt} \equiv \frac{dy}{dt} \tag{S.1}$$

and similar initial positions $x_0 = y_0$ (according to some coordinate systems) at time $t = 0$, then after time T one generally has

$$y(T) = x(T) + 2\pi R \times n \tag{S.2}$$

for some integer $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ because the $x(y)$ path may wrap around the circle n times while the $y(t)$ path may not wrap. For example, the two paths depicted below have same (constant) velocities and begin at $y_0 = x_0$ but end at $y(T) = x(T) + 2\pi R \times 2$:



It is easy to see that the paths $x(t)$ (modulo $2\pi R$) and $y(t)$ (modulo nothing) are in one-to-one correspondence with each other, provided we restrict the initial point y_0 of the particle on the infinite line to a particular interval of length $L = 2\pi R$, say $0 \leq y_0 < 2\pi R$. Consequently, in the path integral for the particle on the circle

$$\int\limits_{x(t=0)=x_0 \,(\mathrm{mod}\, L)}^{x(t=T)=x' \,(\mathrm{mod}\, L)} \mathcal{D}'[x(t) \,(\mathrm{mod}\, L)] \;=\; \sum_{n=-\infty}^{+\infty} \int\limits_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)]\,. \tag{S.3}$$

Furthermore, in the absence of potential energy, the circle path $x(t) \,(\mathrm{mod}\, L)$ and the corresponding ∞ line path $y(t)$ have equal actions

$$S[x(t) \,(\mathrm{mod}\, L)] \;=\; S[y(t)] \;=\int\limits_0^T dt\, \left[\tfrac{M}{2}\dot{x}^2 \;=\; \tfrac{M}{2}\dot{y}^2\right], \tag{S.4}$$

and therefore

$$\begin{aligned} U_{\text{circle}}(x'; x_0) \;&=\; \int\limits_{x(t=0)=x_0 \,(\mathrm{mod}\, L)}^{x(t=T)=x' \,(\mathrm{mod}\, L)} \mathcal{D}'[x(t) \,(\mathrm{mod}\, L)]\, e^{iS[x(t)\,(\mathrm{mod}\, L)]/\hbar} \\ &=\; \sum_{n=-\infty}^{+\infty} \int\limits_{y(t=0)=x_0}^{y(t=T)=x'+nL} \mathcal{D}'[y(t)]\, e^{iS[y(t)]/\hbar} \\ &=\; \sum_{n=-\infty}^{+\infty} U_{\infty\,\text{line}}(y' = x' + nL; y_0 = x_0). \end{aligned} \tag{1}$$

$\mathcal{Q.E.D.}$

Problem **3**(b):

For a free particle living on an infinite line the evolution kernel is given by

$$U_{\infty\,\text{line}}(y'; y_0) \;=\; \sqrt{\frac{M}{2\pi i\hbar T}} \times \exp\left(\frac{i}{\hbar}S_{\text{classical}} \;=\; \frac{i}{\hbar}\,\frac{M(x'-x_0)^2}{2T}\right), \tag{3}$$

hence according to eq. (1), a particle on a circle has kernel

$$U_{\text{circle}}(x'; x_0) \;=\; \sqrt{\frac{M}{2\pi i\hbar T}} \times \sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T}\,(x'-x_0+nL)^2\right). \tag{S.5}$$

To evaluate this sum, we use Poisson re-summation formula (2), which gives

$$\sum_{n=-\infty}^{+\infty} \exp\left(\frac{iM}{2\hbar T}(x'-x_0+nL)^2\right) \;=\; \sum_{\ell=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\nu\, \exp\left(\frac{iM}{2\hbar T}(x'-x_0+\nu L)^2\right)\times e^{2\pi i\ell\nu}. \tag{S.6}$$

Rearranging the exponential, we have

$$\frac{iM}{2\hbar T}(x'-x_0+\nu L)^2 + 2\pi i\ell\nu \;=\; \frac{iML^2}{2\hbar T}\left(\nu + \frac{x'-x_0}{L} + \frac{2\pi\ell\hbar T}{ML^2}\right) - 2\pi i\ell\,\frac{x'-x_0}{L} - \frac{i\hbar T(2\pi\ell)^2}{ML^2}, \tag{S.7}$$

and therefore

$$\int_{-\infty}^{+\infty} d\nu\, \exp\left(\frac{iM}{2\hbar T}(x'-x_0+\nu L)^2\right)\times e^{2\pi i\ell\nu} \;=\; \sqrt{\frac{2\pi i\hbar T}{ML^2}}\times\exp\left(-2\pi i\ell\,\frac{x'-x_0}{L} - \frac{(2\pi\ell)^2 i\hbar T}{ML^2}\right). \tag{S.8}$$

Consequently,

$$\begin{aligned} U_{\rm circle}(x';x_0) \;&=\; \sqrt{\frac{M}{2\pi i\hbar T}}\times\sqrt{\frac{2\pi i\hbar T}{ML^2}}\times\sum_{\ell=-\infty}^{+\infty}\exp\left(-2\pi i\ell\,\frac{x'-x_0}{L} - \frac{(2\pi\ell)^2 i\hbar T}{ML^2)}\right) \\ &=\; \frac{1}{L}\sum_{\ell=-\infty}^{+\infty} e^{ip(x'-x_0)/\hbar}\times e^{-iTE/\hbar} \end{aligned} \tag{S.9}$$

where

$$p \;=\; -\frac{2\pi\hbar\ell}{L} \;=\; -\frac{\hbar\ell}{R} \quad\text{and}\quad E \;=\; \frac{p^2}{2M}\,. \tag{S.10}$$

Problem **3**(c): This is obvious from eqs. (S.9) and (S.10).