Problem **2**:

Ward–Takahashi identity for the scalar QED is similar to same identity for the ordinary (fermionic) QED and has a similar diagrammatic proof. In both cases, we begin with a **Lemma:** Consider a generic Feynman diagram and select a charged-field line⋆ that begins with an incoming external leg of momentum p and ends with an outgoing external leg of momentum p'. Regardless of any other external legs ir internal propagators of the diagram, let $S(p',p)$ denote the corresponding Feynman amplitude. Now consider all possible ways to attach an additional external photonic leg of momentum k to this particular line and total the corresponding amplitudes; let $S^\mu(k;p'+k,p)$ denote this total. To be precise, let the amplitudes $S(p',p)$ and $S^\mu(k;p'+k,p)$ be amputated with respect to external photonic legs but un-amputated WRT the charged-field legs. The lemma says that

$$k_\mu S^\mu(k;p'+k,p) \;=\; eS(p'+k,p+k) \;-\; eS(p',p). \tag{S.1}$$

Notice that it is enough to prove this lemma holds *before* integrating over momenta of the internal photonic propagators. In other words, we may treat all the photonic lines attached to the charged line in question as external. Indeed, once we prove the lemma (S.1) under this restriction, allowing for internal photonic lines means simply multiplying each side of eq. (S.1) by exactly the same photonic propagators and integrating over exactly the same loop momenta. Clearly, if eq. (S.1) holds before such integration, then it automatically holds afterwards as well.

Once this Lemma is proven, the next step is to consider all possible attachments of an external photon to a closed charged-field loop and show that the resulting net amplitude satisfies $k_\mu S^\mu = 0$ after integrating over the loop momentum. For both scalar and fermionic versions of QED, this result follows from the Lemma (S.1) *provided the regularization scheme allows to shift the integration variable* $p \to p' = p + \text{const}$. This argument is explained in the textbook in sufficient detail — there is no difference between fermionic and scalar QEDs on this issue — so there is no need to repeat it here.

⋆ A scalar line for the scalar QED or a fermionic line for the ordinary QED.

Finally, we should consider a diagram with both through lines and loops and total all possible ways of adding one more photon. Given the above, the net result becomes obvious:

$$\begin{aligned} k_\mu S^\mu(p'_1,\ldots,p'_n,p_1,\ldots,p_n) \;&=\; e\sum_{i=1}^{n}\Big[S(p'_1,\ldots,p'_n,p_1,\ldots,(p_i+k),\ldots,p_n) \\ &\qquad -\; S(p'_1,\ldots,(p'_i-k),\ldots,p'_n,p_1,\ldots,p_n)\Big] \end{aligned} \tag{S.2}$$

and that's exactly what the Ward–Takahashi identity says. $\mathcal{Q.E.D.}$

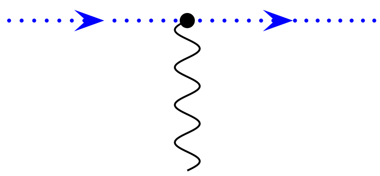
⋆ ⋆ ⋆

The real exercise here is to prove the Lemma (S.1) for the scalar QED; similar to the fermionic QED discussed in the textbook, we shall proceed by induction in the number n of vertices on the selected charged line. The induction base for $n = 0$ has S_0 being a simple propagator,

$$iS_0(p' = p) \;=\; \text{[diagram]} \;=\; \frac{i}{p^2 - m^2} \tag{S.3}$$

while $S_1^\mu(k; p' = p + k)$ comprises two propagators and one vertex, namely

$$iS_1^\mu(k; p' = p + k) \;=\; \text{[diagram]} \;=\; \frac{i}{p'^2 - m^2} \times ie(pp + p)^\mu \times \frac{i}{p^2 - m^2}\,. \tag{S.4}$$

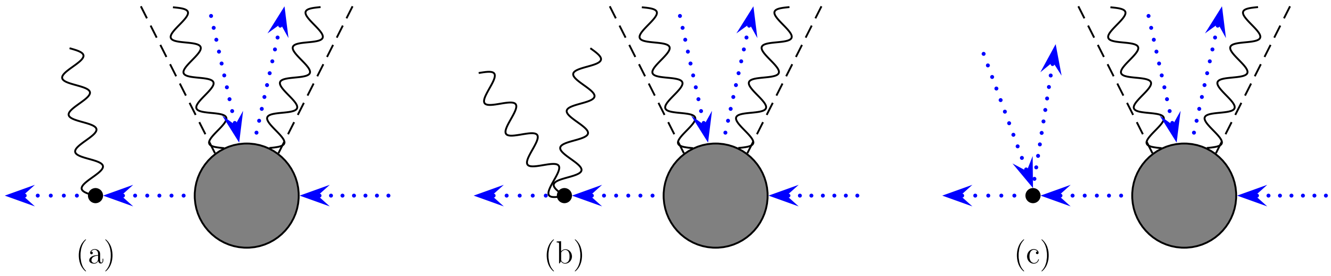


Note that $k = p' - p$ and hence $k_\mu(p + p')^\mu = (p' - p)\cdot(p' + p) = p'^2 - p^2$. Consequently,

$$\begin{aligned} k_\mu \times S_1^\mu(k; p' = p + k, p) \;&=\; k_\mu \times \frac{-e(p + p')}{(p'^2 - m^2)(p^2 - m^2)} \\ &=\; -e\,\frac{p'^2 - p^2}{(p'^2 - m^2)(p^2 - m^2)} \;=\; \frac{e}{p'^2 - m^2} \;-\; \frac{e}{p^2 - m^2} \\ &=\; eS_0(p', p + k) \;-\; eS_0(p' - k, p), \end{aligned} \tag{S.5}$$

in perfect agreement with the Lemma (S.1).

Now suppose any scalar line with n vertices satisfies the Lemma and consider a line with one additional vertex. There are three types of vertices in scalar QED, hence three possibilities:

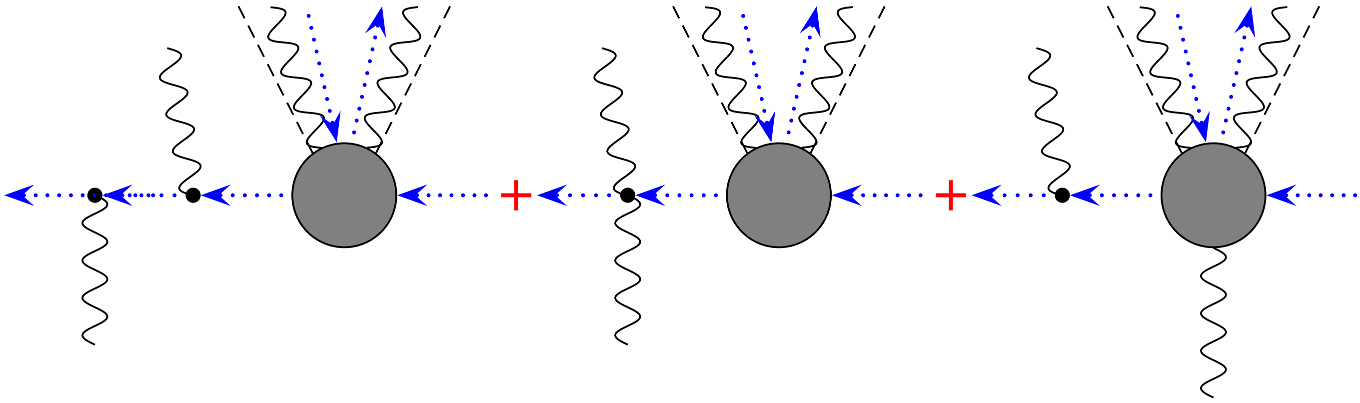


(S.6)

In the first case (a) where the $(n+1)^{\underline{\text{st}}}$ vertex connects the charged line to one photon, the $(n+1)$–vertex amplitude is given by

$$S_{n+1}(p',p) \;=\; \frac{i}{p'^2-m^2}\times ie(p'+(p'-q))^{\nu}\times S_n(p'-q,p). \tag{S.7}$$

To verify the Lemma, we need to attach one more photon to the charged line, and in case (a) there are three possibilities, namely



(S.8)

which total up to amplitude

$$\begin{aligned} S^{\mu}_{n+2}(k;p'+k,p) \;=\; & \frac{i}{(p'+k)^2-m^2}\times ie(2p'+k)^{\mu}\times\frac{i}{p'^2-m^2}\times ie(2p'-q)^{\nu}\times S_n(p'-q,p)\\ &+\;\frac{i}{(p'+k)^2-m^2}\times 2ie^2g^{\mu\nu}\times S_n(p'-q,p)\\ &+\;\frac{i}{(p'+k)^2-m^2}\times ie(2p'+2k-q)^{\nu}\times S\mu_{n+1}(k;p'+k-q,p). \end{aligned} \tag{S.9}$$

Multiplying this amplitude by k_μ, we obtain

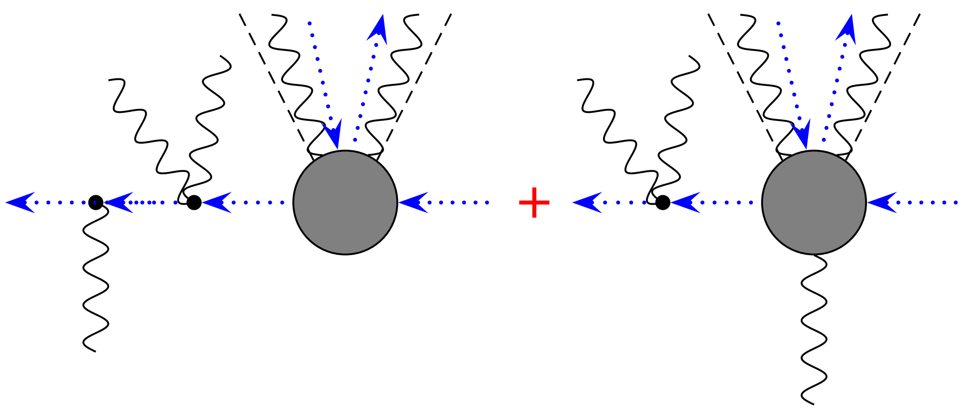
$$\begin{aligned}
k_\mu \times S^\mu_{n+2}(k;p'+k,p) = e^2 &\left[\frac{(2p'k+k^2)\,(2p'-q)^\nu}{((p'+k)^2-m^2)(p'^2-m^2)} \;-\; \frac{2k^\nu}{(p'+k)^2-m^2}\right] \times S_n(p'-q,p)\\
&-\; e\,\frac{(2p'+2k-q)^\nu}{(p'+k)^2-m^2} \times k_\mu S^\mu_{n+1}(k;p'+k-q,p)\\
= e^2 &\left[\frac{(2p'-q)^\nu}{p'^2-m^2} \;-\; \frac{(2p'+2k-q)^\nu}{(p'+k)^2-m^2}\right] \times S_n(p'-q,p)\\
&-\; e\,\frac{(2p'+2k-q)^\nu}{(p'+k)^2-m^2} \times \Big[eS_n(p'+k-q,p+k) - eS_n(p'-q,p)\Big]^{\text{by induction}}_{\text{assumption}}\\
= &-e^2\,\frac{(2p'+2k-q)^\nu}{(p'+k)^2-m^2} \times S_n(p'+k-q,p+k)\\
&+\; e^2\,\frac{(2p'-q)^\nu}{p'^2-m^2} \times S_n(p'-q,p)\\
&\langle\!\langle \text{by eq. (S.7)} \rangle\!\rangle\\
= e&S_{n+1}(p'+k,p+k) \;-\; eS_{n+1}(p',p),
\end{aligned} \tag{S.10}$$

which affirms the Lemma (S.1) for the $n+1$ vertices.

The other two diagrams (S.6) turn out to be simpler. In case (b) where the $(n+1)^{\underline{\text{st}}}$ vertex is a 'seagull' connecting the charged line to two photon lines at once, the $(n+1)$–vertex amplitude is

$$S_{n+1}(p',p) \;=\; \frac{i}{p'^2-m^2} \times 2ie^2 g^{\nu\lambda} \times S_n(p'-q,p) \tag{S.11}$$

where $q = q_1 + q_2$ is the net momentum of the two photons. Again, to verify the Lemma we need to attach one more photon, and this time, there are only two possibilities, namely



(S.12)

which give rise to amplitude

$$S^{\mu}_{n+2}(k;p'+k,p) \;=\; \frac{i}{(p'+k)^2-m^2}\times ie(2p'+k)^{\mu}\times\frac{i}{p'^2-m^2}\times 2ie^2g^{\nu\lambda}\times S_n(p'-q,p) \\ +\;\frac{i}{(p'+k)^2-m^2}\times 2ie^2g^{\nu\lambda}\times S^{\mu}_{n+1}(k;p'+k-q,p). \tag{S.13}$$

Again, multiplying this amplitude by k_μ we find

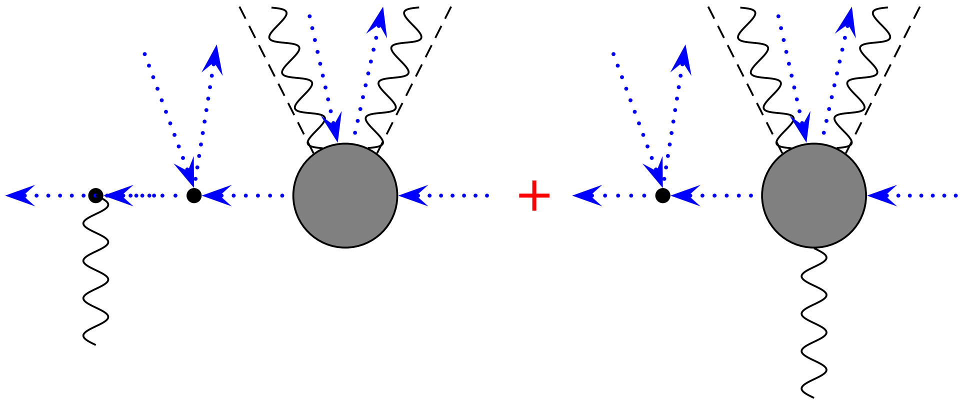
$$\begin{aligned} k_\mu\times S^{\mu}_{n+2}(k;p'+k,p) \;&=\; \frac{k_\mu(2p'-k)^{\mu} \;=\; (p'-k)^2-p'^2}{(p'^2-m^2)((p'+k)^2-m^2)}\times 2e^3g^{\nu\lambda}\times S_n(p'-q,p) \\ &\quad -\;\frac{1}{(p'+k)^2-m^2}\times 2e^2g^{\nu\lambda}\times k_\mu S^{\mu}_{n+1}(k;p'+k-q,p) \\ &=\; \left[\frac{1}{p'^2-m^2}\;-\;\frac{1}{(p'+k)^2-m^2}\right]\times 2e^2g^{\nu\lambda}\times eS_n(p'-q,p) \\ &\quad -\;\frac{1}{(p'+k)^2-m^2}\times 2e^2g^{\nu\lambda}\times\left[eS_n(p'_k-q,p+k)\;-\;eS_n(p'-q,p)\right] \\ &=\; -e\times\frac{1}{(p'+k)^2-m^2}\times 2e^2g^{\nu\lambda}\times S_n(p'-q+k,p+k) \\ &\quad +\;e\times\frac{i}{p'^2-m^2}\times 2e^2g^{\nu\lambda}\times S_n(p'-q,p) \\ &\langle\!\langle\text{by eq. (S.13)}\rangle\!\rangle \\ &=\; +eS_{n+1}(p'+k,p+k)\;-\;eS_{n+1}(p',p), \end{aligned} \tag{S.14}$$

in full agreement with the Lemma (S.1).

Finally, in case (c) where the $(n+1)^{\underline{\text{st}}}$ vertex connects the charged scalar line in question to another scalar line, we have

$$S_{n+1}(p',p) \;=\; \frac{i}{p'^2-m^2}\times -i\lambda\times S_n(p'-q,p) \tag{S.15}$$

where $q=q_{\text{in}}-q_{\text{out}}$ is the net momentum transfer from the other scalar line into the one we are interested in. Again, we need to attach an extra photon to the diagram (c), which gives us two possibilities



(S.16)

and hence amplitude

$$
\begin{aligned}
S^{\mu}_{n+2}(k;p'+k,p) \;&=\; \frac{i}{(p'+k)^2-m^2}\times ie(2p'+k)^{\mu}\times\frac{i}{p'^2-m^2}\times -i\lambda\times S_n(p'-q,p)\\
&+\;\frac{i}{(p'+k)^2-m^2}\times -i\lambda\times S^{\mu}_{n+1}(k;p'+k-q,p).
\end{aligned}
\tag{S.17}
$$

Again, multiplying by the k_μ gives us

$$
\begin{aligned}
k_\mu\times S^{\mu}_{n+2}(k;p'+k,p) \;&=\; \frac{k_\mu(2p'+k)^{\mu}\;=\;(p'+k)^2-p'^2}{(p'^2-m^2)((p'+k)^2-m^2)}\times -e\lambda\times S_n(p'-q,p)\\
&+\;\frac{1}{(p'+k)^2-m^2}\times\lambda\times k_\mu S^{\mu}_{n+1}(k;p'+k-q,p)\\
&=\;\left[\frac{1}{p'^2-m^2}\;-\;\frac{1}{(p'+k)^2-m^2}\right]\times -e\lambda S_n(p'-q,p)\\
&+\;\frac{1}{(p'+k)^2-m^2}\times\lambda\times\Big[=\;eS_n(p'+k-q,p+k)\;-\;eS_n(p'-q,p)\Big]\\
&=\;e\times\frac{1}{(p'+k)^2-m^2}\times\lambda\times S_n(p'-q+k,p+k)\\
&\quad-\;e\times\frac{1}{p'^2-m^2}\times\lambda\times S_n(p'-q,p)\\
&\langle\!\langle\text{by eq. (S.15)}\rangle\!\rangle\\
&=\;eS_{n+1}(p'+k,p+k)\;-\;eS_{n+1}(p',p),
\end{aligned}
\tag{S.18}
$$

in full agreement with the Lemma (S.1).

Altogether, assuming Lemma (S.1) holds for all n-vertex scalar lines, we have proven that is also holds for all $(n+1)$–vertex scalar lines. Hence, by induction, the Lemma must hold true for all scalar lines traversing all possible Feynman graphs of the scalar QED. And as we saw in the beginning of this solution, this Lemma is all we need to prove the Ward–Takahashi identity. $\mathcal{Q.E.D}$.

Problem **3**:

Let us start with a very general QFT where the EM field $A_\mu(x)$ couples to all kinds of charged fields (scalar, fermionic, or even charged vector fields), which may in turn couple to all kinds of neutral fields besides the EM. To simplify notations, let $\varphi_j(x)$ stand for all the non-EM fields of the theory, and consider a correlation function

$$S^{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n;y_1,\ldots,y_m) \;=\; \langle\Omega|\,\mathbf{T}\,A^{\mu_1}(x_1)\cdots A^{\mu_n}(x_n)\,\varphi_1(y_1)\cdots\varphi_m(y_m)\,|\Omega\rangle\,, \tag{S.19}$$

amputated with respect to the EM fields but not the other fields. Then, arguing exactly as I argued in class for the ordinary QED, we can use the path-integral formalism to show that

$$\begin{aligned}\frac{\partial}{\partial x_1^{\mu_1}}S^{\mu_1,\ldots,\mu_n}(x_1,\ldots,x_n;y_1,\ldots,y_m) \;&=\; S^{\mu_2,\ldots,\mu_n}(x_2,\ldots,x_n;y_1,\ldots,y_m)\,\times\\ &\qquad\times\sum_{j=1}^{m} i\delta^{(4)}(x_1-y_j)\times\mathrm{charge}[\varphi_j].\end{aligned} \tag{S.20}$$

Note that only the charged fields enter the right hand side of this formula — the neutral fields (both the neutral $\varphi_j(y_j)$ and the $A^{\mu_2}(x_2),\ldots,A^{\mu_n}(x_n)$) simply 'go along for the ride'.

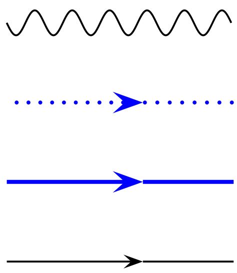
In momentum space, eq. (S.20) becomes the Ward–Takahashi identity in its most general form. Treating all external particles' momenta and charges as incoming, we have

$$k_1^{\mu_1}\times S_{\mu_1,\ldots,\mu_n}(k_1,\ldots,k_n;p_1,\ldots,p_n) \;=\; -\sum_{j=1}^{m}\mathrm{charge}[j]\times S_{\mu_2,\ldots,\mu_n}(k_2,\ldots,k_n;p_1,\ldots,(p_j+k_1),\ldots,p_n). \tag{S.21}$$

Or equivalently, if we prefer to treat some eternal line as incoming and others as outgoing, then in obvious notations

$$
\begin{aligned}
k_1^\mu \times S_{\mu,\ldots}(k_1,\ldots,k_n;p'_1,\ldots,p'_{m'};p_1,\ldots,p_m) \;&= \\
=\; -\sum_{j=1}^{m} &\text{charge}[j] \times S_{\ldots}(k_2,\ldots,k_n;p'_1,\ldots,p'_{m'};p_1,\ldots,(p_j+k_1),\ldots,p_m) \\
+\sum_{j=1}^{m'} &\text{charge}[j] \times S_{\ldots}(k_2,\ldots,k_n;p'_1,\ldots,(p'_j-k_1),\ldots,p'_{m'};p_1,\ldots,p_m).
\end{aligned}
\tag{S.22}
$$

Now let us focus on the specific theory in question. We have four distinct fields here, hence four types of propagators:



photon A^μ,

charged scalar ϕ,

charged Dirac fermion ψ,

neutral Dirac fermion χ.

(S.23)

Consequently, a generic amplitude may have several arrowless photonic lines, a bunch of incoming and outgoing lines, of three different types. Instead of giving all these lines distinct notations and going crazy trying to keep kinds of momenta straight, let us consolidate: Let p_j (p'_j) stand for the incoming (outgoing) momenta of all the charged lines, scalar or fermionic alike. For the neutral fermions, we need a separate letter, so let's use q and q'. Then, a typical amplitude can be written as

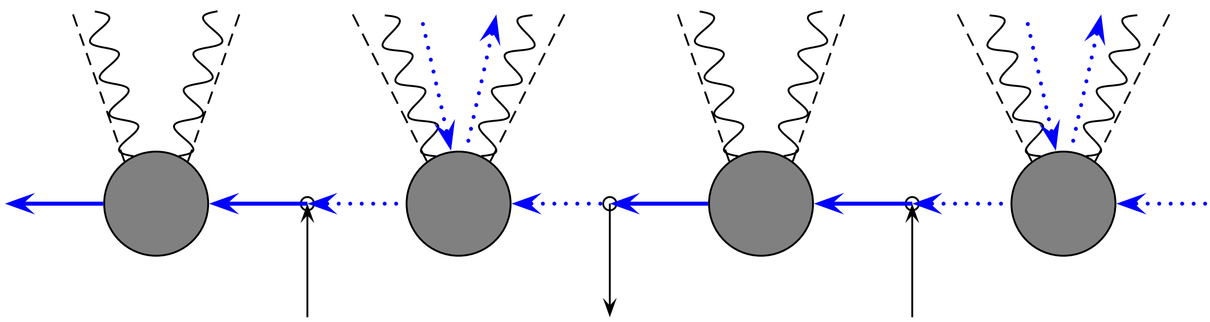
$$
S^{\mu,\cdots}(k_1,k_2,\ldots;p'_1,\ldots,p'_m;p_1,\ldots,p_m;q',\ldots,q,\ldots) \tag{S.24}
$$

where we suppress all kinds of details un-important for the present purposes: The Dirac indices of all the fermions, the vector indices of all the photons except the first, how many neutral particles are involved (if any), or which of the charged particles are scalars and which are fermions. Then in these notations, the Ward–Takahashi identity looks (almost) exactly as in QED:

$$\begin{aligned} k_1^\mu \times S_{\mu,\ldots}(k_1, k_2, \ldots; p'_1, \ldots, p'_m; p_1, \ldots, p_m; q', \ldots, q, \ldots) \;&= \\ = \; e\sum_{j=1}^{m} S_{\ldots}(k_2, \ldots; p'_1, \ldots, p'_m; p_1, \ldots, (p_j + k_1), \ldots, p_m; q', \ldots, q, \ldots)& \\ -e\sum_{j=1}^{m} S_{\ldots}(k_2, \ldots; p'_1, \ldots, (p'_j - k_1), \ldots, p'_m; p_1, \ldots, p_m; q', \ldots, q, \ldots),& \end{aligned} \tag{S.25}$$

assuming the ϕ and ψ fields have electric charge $-e$.

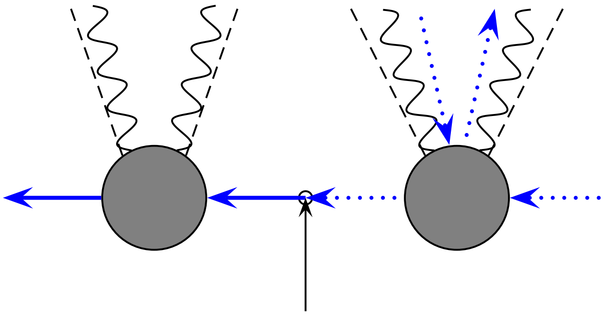
The diagrammatic proof of this identity is based on a Lemma concerning a single charged line going through some complicated Feynman diagram. Formally, the lemma looks just like eq. (S.1) of the previous problem, but now we have to account for charged lines having both fermionic and scalar segments, for example



(S.26)

Each individual fermionic segment works exactly as a fermionic line in ordinary QED and satisfies the Lemma for exactly the same reason. Likewise, each scalar segment works like a charged scalar line in scalar QED and hence also satisfies the Lemma. So all we need to do now is to combine the segments together.

Consider a two-segment charged line

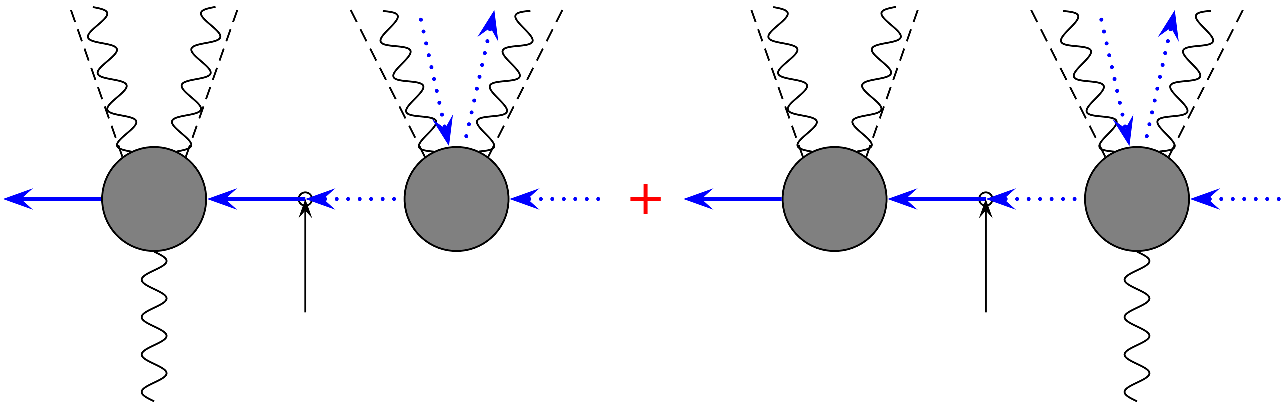


(S.27)

which gives rise to amplitude

$$iS(p';p;q;\text{other}) \;=\; iS_\psi(p';\tilde{p}+q;\text{other}) \times i\gamma u(q) \times iS_\phi(\tilde{p};p;\text{other}). \qquad \text{(S.28)}$$

Now let's add a photon. The new photon can be attached to either segment, thus



(S.29)

and hence amplitude

$$\begin{aligned} iS^\mu(k;p'+k;p;q;\text{other}) \;&=\; iS^\mu_\psi(k;p'+k;\tilde{p}+q;\text{other}) \times i\gamma u(q) \times iS_\phi(\tilde{p};p;\text{other}) \\ &\quad+\; iS_\psi(p'+k;\tilde{p}+q+k;\text{other}) \times i\gamma u(q) \times iS^\mu_\phi(k;\tilde{p}+k;p;\text{other}). \end{aligned} \qquad \text{(S.30)}$$

For each segment, we have

$$\begin{aligned} k_\mu \times S^\mu_\psi(k;p'+k;\tilde{p}+q;\text{other}) \;&=\; eS_\psi(p'+k;\tilde{p}+q+k;\text{other}) \;-\; eS_\psi(p';\tilde{p}+q;\text{other}), \\ k_\mu \times S^\mu_\phi(k;\tilde{p}+k;\tilde{p};\text{other}) \;&=\; eS_\psi(\tilde{p}+k;\tilde{p}+k;\text{other}) \;-\; eS_\psi(\tilde{p};\tilde{p};\text{other}), \end{aligned} \qquad \text{(S.31)}$$

hence the two-segment amplitude (S.30) satisfies

$$\begin{aligned}
k_\mu \times iS^\mu(k; p' + k; p; q; \text{other}) \;&=\; ie\Big[S_\psi(p' + k; \tilde{p} + q + k; \text{other}) \;-\; S_\psi(p'; \tilde{p} + q; \text{other})\Big] \times \\
&\qquad\qquad \times i\gamma u(q) \times iS_\phi(\tilde{p}; p; \text{other}) \\
&\quad +\; iS_\psi(p' + k; \tilde{p} + q + k; \text{other}) \times i\gamma u(q) \times \\
&\qquad\qquad \times ie\Big[S_\psi(\tilde{p} + k; \tilde{p} + k; \text{other}) \;-\; S_\psi(\tilde{p}; \tilde{p}; \text{other})\Big] \\
&=\; e \times iS_\psi(p' + k; \tilde{p} + q + k; \text{other}) \times i\gamma u(q) \times S_\psi(\tilde{p} + k; \tilde{p} + k; \text{other}) \\
&\quad -\, e \times iS_\psi(p' + k; \tilde{p} + q + k; \text{other}) i\gamma u(q) \times iS_\phi(\tilde{p}; p; \text{other}) \\
&\langle !\langle \text{by eq. (S.28)} \rangle\!\rangle \\
&=\; e \times iS(p' + k; p + k; q; \text{other}) \;-\, e \times iS(p'; p; q; \text{other}).
\end{aligned} \tag{S.32}$$

In other words, we have just proven that if both fermionic and scalar charged line segments satisfy the Lemma, then the two-segment charged line also satisfies the Lemma. Furthermore, the same argument applies to charged lines comprising any number of segments — this should be obvious, but please work it out — and this proves the Lemma for all kind of open charged lines in the theory in question.

And once the Lemma is proven, we proceed exactly as in the textbook and show that $k_\mu S^\mu(k; \cdots)$ vanishes for the closed loops of charged lines, and hence generic amplitudes containing both closed loops and open lines satisfy the Ward identity (S.25).