

Problem 1(a):

Let us start with the superficial degree of divergence. Scalar QED is a purely bosonic theory where all propagators behave as  $1/q^2$  at large momenta. On the other hand, the  $\gamma\phi\bar{\phi}$  vertex of the scalar QED behaves as  $q^{+1}$ , assuming at least one of the scalar propagators belongs to a loop. Hence, the superficial degree of divergence of a Feynman diagram is given by

$$\mathcal{S} = 4L - 2P + V_3 \tag{S.1}$$

where  $P$  is the net number of propagators — scalar or vector — and  $V_3$  is the number of tri-valent vertices. Let us also introduce  $V_4$  — the net number of four-valent vertices,  $\gamma\gamma\phi\bar{\phi}$  or  $\phi\phi\bar{\phi}\bar{\phi}$ , and  $E$  — the net number of external legs. Simple counting of line ends gives us

$$2P + E = 3V_3 + 4V_4 \tag{S.2}$$

while the Euler theorem says

$$L - P + V_3 + V_4 = 1. \tag{S.3}$$

Combining these two formulæ, we have

$$4L - 2P = 4 - E - V_3 \tag{S.4}$$

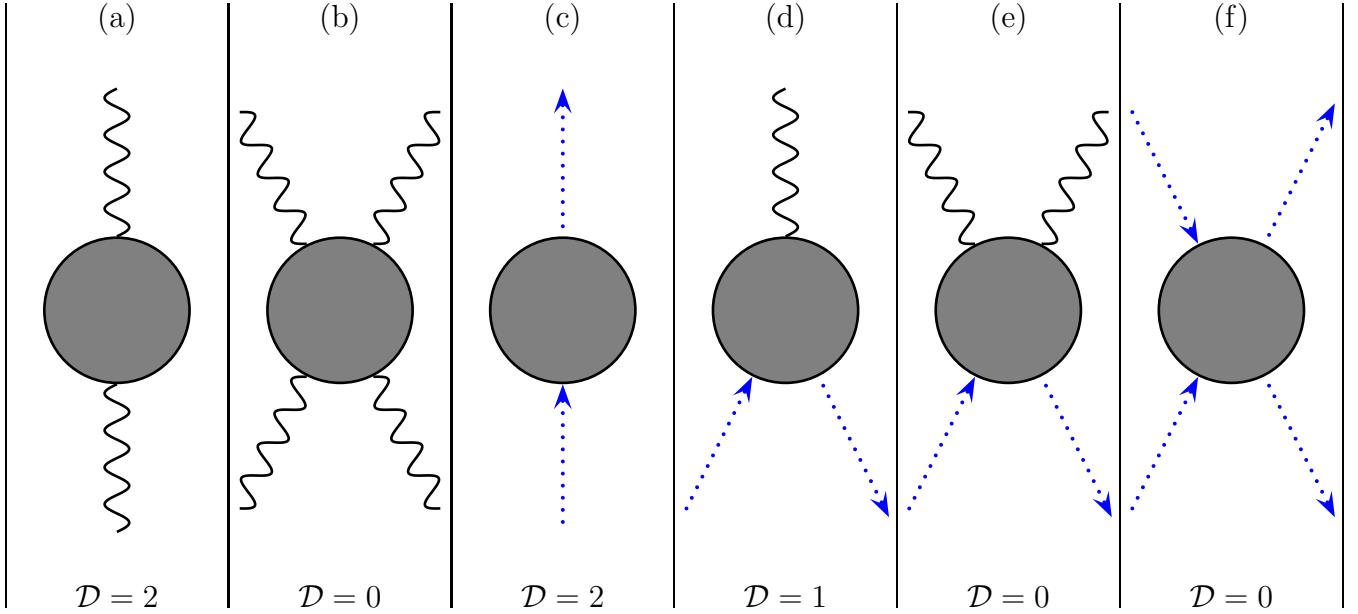
and therefore

$$\mathcal{D} = 4 - E \tag{S.5}$$

regardless of the actual numbers of loops, propagators, or vertices.

In light of eq. (S.5), all superficially divergent amplitudes must have  $E \leq 4$  external legs. Among those external legs, we must have equal numbers of incoming and outgoing charge arrows, thus  $E_\phi = E_{\bar{\phi}} = 0, 1, 2$  only. Furthermore, for  $E_\phi = 0$ , the number of photonic legs  $E_\gamma$  must be even because of the charge conjugation symmetry of the theory. Finally, we may disregard the  $E = 0$

vacuum energy “amplitude”: it diverges, but it’s literally disconnected from the rest of the theory. Between these restrictions, we have six superficially divergent 1PI amplitudes, namely



Now let us consider the actual divergences of these amplitudes.

- (a) The two-photon amplitude  $-i\Sigma^{\mu\nu}(k)$  behaves exactly as in the ordinary QED. By Lorentz symmetry, we must have

$$\Sigma^{\mu\nu}(k) = g^{\mu\nu} \times \mathcal{R}(k^2) + k^\mu k^\nu \times \Pi(k^2) \quad (\text{S.6})$$

for some scalar functions  $\mathcal{R}(k^2)$  and  $\Pi(k^2)$ , and then the Ward–Takahashi identity requires  $k_\mu \times \Sigma^{\mu\nu} = 0$ , hence  $\mathcal{R} = k^2 \times \Pi$  and therefore

$$\Sigma^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu) \times \Pi(k^2). \quad (\text{S.7})$$

This factorization lowers the superficial degree of divergence of the two-photon amplitude by two units; consequently,  $\Pi(k^2)$  diverges logarithmically rather than quadratically, and the divergence itself is momentum-independent, thus

$$\Pi(k^2) = [O(\log \Lambda^2) \text{ constant}] + \text{finite function}(k^2) \quad (\text{S.8})$$

Hence, to cancel this divergence we need just one counterterm vertex with a fixed momentum



Indeed, by Lorentz symmetry we must have

$$G^\mu(p', p) = (p' + p)^\mu \times A + (k = p' - p)^\mu \times B \quad (\text{S.14})$$

where  $A$  and  $B$  are scalar functions of the Lorentz invariants  $p^2$ ,  $p'^2$ , and  $k^2$ . Because of the extra factors of momentum in this formula,  $A$  and  $B$  are only logarithmically divergent, and hence their infinite parts are momentum-independent, thus

$$\begin{aligned} A &= [O(\log \Lambda^2) \text{ constant}] + \text{finite function}(p^2, p'^2, k^2), \\ B &= [O(\log \Lambda^2) \text{ constant}] + \text{finite function}(p^2, p'^2, k^2), \end{aligned} \quad (\text{S.15})$$

Furthermore, we shall see in problem 1(d) that  $G^\mu(p', p)$  satisfies a Ward–Takahashi identity

$$k_\mu G^\mu(p', p) = (p'^2 - M^2 - \Sigma(p'^2)) - (p^2 - M^2 - \Sigma(p^2)) \quad (\text{S.16})$$

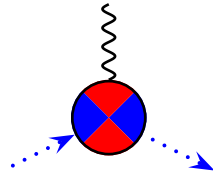
where the right hand side vanishes when  $p'^2 - p^2$ . On the other hand,

$$k_\mu G^\mu(p', p) = (p'^2 - p^2) \times A + k^2 \times B,$$

which means that  $B$  must vanish when  $p'^2 - p^2$ . And since the infinite part of  $B$  is momentum independent, it follows that  $B$  cannot diverge at all. Therefore

$$G^\mu(p', p) = O(\log \Lambda^2) \times (p' + p)^\mu + \text{finite function}(p', p), \quad (\text{S.17})$$

and the divergence here can be canceled by a single counterterm, namely



$= ie(p' + p)^\mu \times \delta_1.$

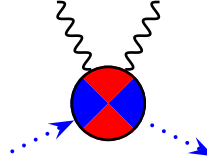
$(\text{S.18})$

- (e) The two-scalars-two-photons 1PI amplitude  $2ie^2 G^{\mu\nu}(k_1, k_2; p', p)$  is also peculiar to the scalar QED. In this case, the superficial degree of divergence  $\mathcal{D} = 0$  tells the truth: The amplitude

diverges logarithmically, because it has no reason to do otherwise. As usual, the infinite part of the log-divergent amplitude is momentum-independent, thus

$$G^{\mu\nu}(k_1, k_2; p', p) = O(\log \Lambda^2) \times \text{constant}^{\mu\nu} + \text{finite function}(k_1, k_2; p', p), \quad (\text{S.19})$$

where by Lorentz symmetry,  $\text{constant}^{\mu\nu} \propto g^{\mu\nu}$ . Consequently, the divergence here can be canceled by a single momentum-independent counterterm, namely



$$= 2ie^2 g^{\mu\nu} \times \delta'_1. \quad (\text{S.20})$$

(f) Finally, the four-scalar 1PI amplitude  $i\mathcal{V}(p_1, p_2, p'_1, p'_2)$  has superficial degree of divergence  $\mathcal{D} = 0$  and indeed diverges logarithmically. As usual, the infinite part of the log-divergent amplitude is momentum-independent, thus

$$\mathcal{V}(p_1, p_2, p'_1, p'_2) = [O(\log \Lambda^2) \text{ constant}] + \text{finite function}(p_1, p_2, p'_1, p'_2), \quad (\text{S.21})$$

which needs a single momentum-independent counterterm to cancel, namely



$$= -i \times \delta_\lambda. \quad (\text{S.22})$$

Similar to the Yukawa theory, this counterterm is needed even in the absence of a tree-level four-scalar coupling.

Problem 1(b):

In part (a) of this problem, we found that scalar QED needs six counterterms, namely  $\delta_3$ ,  $\delta_2$ ,  $\delta_m$ ,  $\delta_1$ ,  $\delta'_1$ , and  $\delta_\lambda$ , *cf.* eqs. (S.9), (S.13), (S.18), (S.20), and (S.22). It is easy to see that all these

counterterm arise from a local Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{terms}}^{\text{counter}} &= -\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \delta_2 \partial^\mu\Phi^*\partial_\mu\Phi - \delta_m \Phi^*\Phi \\ &+ ie\delta_1 A^\mu(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*) + e^2\delta'_1 A^\mu A_\mu\Phi^*\Phi - \frac{1}{4}\delta_\lambda(\Phi^*\Phi)^2.\end{aligned}\tag{S.23}$$

Moreover, the counterterms here are exactly similar to the physical Lagrangian terms

$$\begin{aligned}\mathcal{L}^{\text{phys}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial^\mu\Phi^*\partial_\mu\Phi - M^2\Phi^*\Phi \\ &+ ieA^\mu(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*) + e^2A^\mu A_\mu\Phi^*\Phi - \frac{1}{4}\lambda(\Phi^*\Phi)^2.\end{aligned}\tag{S.24}$$

of the scalar QED, only the coefficients of the various terms are different. Combining the two sets of terms, we end up with the bare Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{bare}} &= \mathcal{L}^{\text{phys}} + \mathcal{L}_{\text{terms}}^{\text{counter}} \\ &= -\frac{1}{4}Z_3 F_{\mu\nu}F^{\mu\nu} + Z_2 \partial^\mu\Phi^*\partial_\mu\Phi - Z_2 M_b^2 \Phi^*\Phi \\ &+ ieZ_1 A^\mu(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*) + e^2 Z_1' A^\mu A_\mu\Phi^*\Phi - \frac{1}{4}Z_2^2 \lambda_b (\Phi^*\Phi)^2 \\ &= -\frac{1}{4}(F_b^{\mu\nu})^2 + \partial^\mu\Phi_b^*\partial_\mu\Phi_b - M_b^2 \Phi_b^*\Phi_b \\ &+ ie_b A_b^\mu(\Phi_b^*\partial_\mu\Phi_b - \Phi_b\partial_\mu\Phi_b^*) + e_b'^2 (A_b^\mu)^2 \Phi_b^*\Phi_b - \frac{1}{4}\lambda_b (\Phi_b^*\Phi_b)^2\end{aligned}\tag{S.25}$$

where

$$\begin{aligned}Z_3 &= 1 + \delta_3, & Z_2 &= 1 + \delta_2, & Z_1 &= 1 + \delta_1, & Z_1' &= 1 + \delta_1', \\ Z_2 M_b^2 &= M^2 + \delta_m, & Z_2^2 \lambda_b &= \lambda + \delta_\lambda, \\ \Phi_b(x) &= \sqrt{Z_2}\Phi(x), & A_b^\mu(x) &= \sqrt{Z_3}A^\mu(x), \\ Z_2\sqrt{Z_3} \times e_b &= Z_1 \times e, & Z_2 Z_3 \times e_b'^2 &= Z_1' \times e^2.\end{aligned}\tag{S.26}$$

By inspection, the bare Lagrangian (S.25) is local, and all the interaction have dimensionless coefficients, which ordinarily implies renormalizability.

However, the gauge theories such as QED or scalar QED are not quite ordinary, and their renormalizability depends crucially on the gauge invariance — otherwise, the Ward identity we

have used in part (a) would not work. But in order to assemble the

$$\partial^\mu \Phi_b^* \partial_\mu \Phi_b + ie_b A_b^\mu (\Phi_b^* \partial_\mu \Phi_b - \Phi_b \partial_\mu \Phi_b^*) + e_b'^2 (A_b^\mu)^2 \Phi_b^* \Phi_b \quad (\text{S.27})$$

terms of the bare Lagrangian into the gauge-invariant combination  $D^\mu \Phi_b^* D_\mu \Phi_b$ , we must have  $e_b'^2 = e_b^2$ , and according to eq. (S.26) this requires

$$Z_1' = \frac{Z_1^2}{Z_2}. \quad (\text{S.28})$$

Furthermore, the way the fields transform under the gauge symmetry

$$\Phi(x) \mapsto \Phi(x) \times e^{i\theta(x)}, \quad eA_\mu(x) \mapsto eA_\mu(x) - \partial_\mu \theta(x) \quad (\text{S.29})$$

should not be subject to renormalization, which means we should have

$$e_b A_b^\mu(x) = e A^\mu(x), \quad (\text{S.30})$$

which in turn requires

$$Z_1' = Z_1 = Z_2. \quad (\text{S.31})$$

In part (d) of this problem (see solutions for set #20), we shall prove that this equality does hold true thanks to Ward identities of the scalar QED.

Problem 1(c):

The renormalization conditions for the  $\delta_2$ ,  $\delta_m$ , and  $\delta_\lambda$  counterterms are similar to those of the  $\lambda\phi^4$  theory:

$$\Sigma_{\text{net}}(p^2) = 0 \quad \text{and} \quad \frac{d\Sigma_{\text{net}}}{dp^2} = 0 \quad \text{for} \quad p^2 = M^2 \quad (\text{S.32})$$

determine the finite parts of the  $\delta_2$  and  $\delta_m$  counterterms, and

$$\mathcal{V}_{\text{net}}(p_1, p_2, p_1', p_2') = -\lambda \quad \text{for} \quad p_1 = p_2 = p_1' = p_2' \quad \text{and} \quad p^2 = M^2 \quad (\text{S.33})$$

determines the finite part of the  $\delta_\lambda$ . Specifically,

$$\delta_2 = \left. \frac{d\Sigma_{\text{loops}}}{dp^2} \right|_{p^2=M^2}, \quad (\text{S.34})$$

$$\delta_m - M^2 \delta_2 = \Sigma_{\text{loops}}(p^2 = M^2), \quad (\text{S.35})$$

$$\delta_\lambda = \mathcal{V}_{\text{loops}}(p_1 = p_2 = p'_1 = p'_2, p^2 = M^2). \quad (\text{S.36})$$

Likewise, for the  $\delta_3$  counterterm, the renormalization condition works exactly as in the ordinary QED:

$$\Pi_{\text{net}}(k^2) = 0 \quad \text{for } k^2 = 0 \quad (\text{S.37})$$

and hence

$$\delta_3 = -\Pi_{\text{loops}}(k^2 = 0). \quad (\text{S.38})$$

Renormalization conditions for the two remaining counterterms,  $\delta_1$  and  $\delta'_1$  are peculiar to the scalar QED and cannot be copied verbatim from another theory. Instead, we generalize from the ordinary QED where the condition for the finite part of  $\delta_1$  reads

$$\Gamma_{\text{net}}^\mu(p', p) = \gamma^\mu \quad \text{for } \not{p}' = \not{p} = m. \quad (\text{S.39})$$

For the scalar QED, we should likewise set  $k = 0 \Rightarrow p' = p$  and put the scalar momentum on shell. Thus, we require

$$G_{\text{net}}^\mu(p', p) = (p' + p)^\mu \quad \text{for } p' = p \text{ and } p^2 = M^2, \quad (\text{S.40})$$

which indeed determines the finite part of the  $\delta_1$  counterterm according to

$$\begin{aligned} eG_{\text{net}}^\mu(p', p) &= e(p' + p)_{\text{tree}}^\mu + eG_{\text{loops}}^\mu(p', p) + e\delta_1 \times (p' + p)^\mu \\ &\Downarrow \\ \delta_1 \times (p' + p)^\mu &= -G_{\text{loops}}^\mu \quad \text{for } p' = p \text{ and } p^2 = M^2. \end{aligned} \quad (\text{S.41})$$

Finally, for the  $\delta'_1$  counterterm, we consider the two-scalar-two-photon 1PI amplitude  $G^{\mu\nu}$  and set  $k_1 = k_2 = 0$ . By Lorentz symmetry, in this limit

$$G_{\text{net}}^{\mu\nu}(0, 0; p' = p) = \mathcal{A}(p^2)_{\text{net}} \times g^{\mu\nu} + \mathcal{B}(p^2)_{\text{net}} \times p^\mu p^\nu, \quad (\text{S.42})$$

and we fix the  $\delta'_1$  counterterm by requiring  $\mathcal{A} = 2$  when  $p' = p$  is on-shell. Indeed, combining the



tree-level, loops, and counterterm contributions the net  $e^2 G^{\mu\nu}$  amplitude, we have

$$e^2 G_{\text{net}}^{\mu\nu}(\text{momenta}) = (2e^2 g^{\mu\nu})_{\text{tree}} + e^2 G_{\text{loops}}^{\mu\nu}(\text{momenta}) + 2e^2 g^{\mu\nu} \times \delta'_1, \quad (\text{S.43})$$

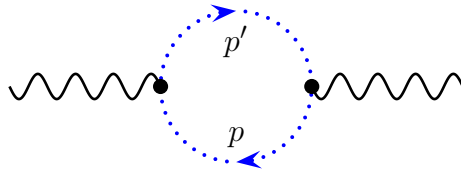
hence for  $k_1 = k_2 = 0$

$$A_{\text{net}}(p^2) = 2_{\text{tree}} + A_{\text{loops}}(p^2) + 2\delta'_1 \implies \delta'_1 = -\frac{1}{2}A_{\text{loops}}(p^2 = M^2). \quad (\text{S.44})$$

Problem 1(d) is delayed till next homework. The solution will appear as part of solutions for set #20.

Problem 2(a):

Let us start with the first diagram



(S.45)

Direct evaluation of the Feynman rules gives us

$$-i\Sigma_{(1)}^{\mu\nu}(k) = \int \frac{d^4p}{(2\pi)^4} ie(p' + p)^\mu \times \frac{i}{p^2 - M^2 + i0} \times ie(p + p')^\nu \times \frac{i}{p'^2 - M^2 + i0} \quad (\text{S.46})$$

where  $p' = p + k$ . As usual, we combine the two denominators using Feynman parameter trick, thus

$$\frac{1}{p^2 - M^2 + i0} \times \frac{1}{(p + k)^2 - M^2 + i0} = \int_0^1 dx \frac{1}{[q^2 - \Delta + i0]^2} \quad (\text{S.47})$$

where

$$q^2 - \Delta = x(p + k)^2 + (1 - x)p^2 - M^2 \quad (\text{S.48})$$

and hence

$$q = p + xk \quad \text{and} \quad \Delta = M^2 - x(1 - x)k^2. \quad (\text{S.49})$$

Next, we shift the integration variable from  $p$  to  $q$ , and this gives us

$$\Sigma_{(1)}^{\mu\nu}(k) = ie^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{(p+p')^\mu(p+p')^\nu}{[q^2 - \Delta + i0]^2} \quad (\text{S.50})$$

where in the numerator

$$\begin{aligned} (p+p')^\mu(p+p')^\nu &= (2q + (1-2x)k)^\mu(2q + (1-2x)k)^\nu \\ &= 4q^\mu q^\nu + (1-2x)^2 k^\mu k^\nu + 2(1-2x)[q^\mu k^\nu + k^\mu q^\nu] \end{aligned} \quad (\text{S.51})$$

Note that the last term on the second line here is odd with respect to  $q$  and hence does not contribute to the  $\int dq$  integral. As to the first term on the second line, in the context of  $\int dq$  integral  $q^\mu q^\nu$  is equivalent to  $g^{\mu\nu} \times q^2/D$ . Altogether, we have

$$\begin{aligned} (p+p')^\mu(p+p')^\nu &\cong \frac{4}{D} q^2 \times g^{\mu\nu} + (1-2x)^2 k^\mu k^\nu \\ &= (k^\mu k^\nu - k^2 g^{\mu\nu}) \times (1-2x)^2 + g^{\mu\nu} \times \left( \frac{4}{D} q^2 + (1-2x)^2 k^2 \right), \end{aligned} \quad (\text{S.52})$$

and consequently

$$\Sigma_{(1)}^{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu) \times \Pi_{(1)}(k^2) + g^{\mu\nu} \times \Xi_{(1)}(k^2) \quad (\text{S.53})$$

where

$$\Pi_{(1)}(k^2) = e^2 \int_0^1 dx (1-2x)^2 \int_{\text{reg}} \frac{d^4q}{(2\pi)^4} \frac{-i}{(q^2 - \Delta + i0)^2} \quad (\text{S.54})$$

and

$$\Xi_{(1)}(k^2) = e^2 \int_0^1 dx i \int_{\text{reg}} \frac{d^4q}{(2\pi)^4} \frac{\frac{4}{D} q^2 + (1-2x)^2 k^2}{(q^2 - \Delta + i0)^2}. \quad (\text{S.55})$$

Our first task is to verify the tensor structure of the two-photon amplitude, so let us focus on the coefficient  $\Xi$  of the wrong tensor. Applying Wick rotation and dimensional regularization to

the momentum integral in eq. (S.55), we calculate

$$\begin{aligned}
& i \int_{\text{reg}} \frac{d^4 q}{(2\pi)^4} \frac{\frac{4}{D} q^2 + (1-2x)^2 k^2}{(q^2 - \Delta + i0)^2} = \\
& = \int_{\text{reg}} \frac{d^4 q_E}{(2\pi)^4} \frac{\frac{4}{D} q_E^2 - (1-2x)^2 k^2}{(q^2 + \Delta)^2} \\
& = \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\frac{4}{D} q^2 - (1-2x)^2 k^2}{(q^2 + \Delta)^2} \\
& = \mu^{4-D} \int_0^\infty dt t \int \frac{d^D q}{(2\pi)^D} \left( \frac{4}{D} q^2 - (1-2x)^2 k^2 \right) e^{-t(q^2 + \Delta)} \\
& = \mu^{4-D} \int_0^\infty dt t e^{-t\Delta} \left( -\frac{4}{D} \frac{\partial}{\partial t} - (1-2x)^2 k^2 \right) \int \frac{d^D q}{(2\pi)^D} e^{-tq^2} \\
& = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \int_0^\infty dt t e^{-t\Delta} \left( -\frac{4}{D} \frac{\partial}{\partial t} - (1-2x)^2 k^2 \right) t^{-D/2} \\
& = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \int_0^\infty dt t e^{-t\Delta} \left( 2t^{-(D/2)-1} - (1-2x)^2 k^2 \times t^{-D/2} \right) \\
& = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \left( 2\Gamma\left(1 - \frac{D}{2}\right) \Delta^{\frac{D}{2}-1} - (1-2x)^2 k^2 \times \Gamma\left(2 - \frac{D}{2}\right) \Delta^{\frac{D}{2}-2} \right) \\
& = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \times \left[ 2\Delta^{\frac{D}{2}-1} + \left(\frac{D}{2} - 1\right) \Delta^{\frac{D}{2}-1} \times (2x-1) \frac{\partial \Delta}{\partial x} \right] \\
& = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \times \frac{\partial}{\partial x} \left( (2x-1) \Delta^{\frac{D}{2}-1} \right), \tag{S.56}
\end{aligned}$$

and consequently

$$\begin{aligned}
\Xi_{(1)}(k^2) &= e^2 \int_0^1 dx \frac{\mu^{4-D}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \frac{\partial}{\partial x} \left( (2x-1) \Delta^{\frac{D}{2}-1} \right) \\
&= \frac{e^2 \mu^{4-D}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \times \left[ \Delta^{\frac{D}{2}-1} \Big|_{x=1} + \Delta^{\frac{D}{2}-1} \Big|_{x=0} \right] = 2(M^2)^{\frac{D}{2}-1} \\
&= \frac{\alpha m^2}{2\pi} \times \Gamma\left(1 - \frac{D}{2}\right) \times \left( \frac{4\pi\mu^2}{M^2} \right)^{2-\frac{D}{2}}.
\end{aligned} \tag{S.57}$$

Note that thanks to  $\Delta(x=1) = \Delta(x=0) = m_0^2$ , the bottom line of eq. (S.57) is independent of the photon's momentum  $k$ . And since the second diagram's contribution  $\Sigma_{(2)}^{\mu\nu}$  is also  $k$ -independent, this allows for the cancellation of the wrong tensor structure of the two-photon amplitude between the two diagrams.

Indeed, for the second diagram

$$\tag{S.58}$$

we have

$$-i\Sigma_{(2)}^{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4} 2ie^2 g^{\mu\nu} \times \frac{i}{p^2 - M^2 + i0} \tag{S.59}$$

which does not depend on the photons' momenta and has wrong tensor structure

$$\Sigma_{(2)}^{\mu\nu} = g^{\mu\nu} \times \Xi_{(2)}. \tag{S.60}$$

To evaluate the coefficient  $\Xi_{(2)}$  of this wrong tensor structure, we continue the loop momentum  $p$

to Euclidean space and then use dimensional regularization, thus

$$\begin{aligned}
\Xi_{(2)} &= \int_{\text{reg}} \frac{d^4 p}{(2\pi)^4} \frac{-2ie^2}{p^2 - M^2 + i0} \\
&= \int_{\text{reg}} \frac{d^4 p_E}{(2\pi)^4} \frac{-2e^2}{p_E^2 + M^2} \\
&= -2e^2 \mu^{4-D} \int \frac{d^D p_E}{(2\pi)^D} \int_0^\infty dt e^{-t(M^2 + p_E^2)} \\
&= -2e^2 \mu^{4-D} \int_0^\infty dt e^{-tM^2} \int \frac{d^D p_E}{(2\pi)^D} e^{-tp_E^2} \\
&= -\frac{2e^2 \mu^{4-D}}{(4\pi)^{D/2}} \int_0^\infty dt e^{-tM^2} t^{-D/2} \\
&= -\frac{2e^2 \mu^{4-D}}{(4\pi)^{D/2}} \times \Gamma\left(1 - \frac{D}{2}\right) (M^2)^{\frac{D}{2}-1} \\
&= -\frac{\alpha m^2}{2\pi} \times \Gamma\left(1 - \frac{D}{2}\right) \times \left(\frac{4\pi\mu^2}{M^2}\right)^{2-\frac{D}{2}}
\end{aligned} \tag{S.61}$$

Comparing this formula to the wrong-tensor contribution (S.57) of the first diagram, we immediately see that they cancel each other,

$$\Xi_{1\text{loop}} = \Xi_{(1)} + \Xi_{(2)} = 0 \tag{S.62}$$

and therefore

$$\Sigma_{1\text{loop}}^{\mu\nu} = (k^\mu k^\nu - k^2 g^{\mu\nu}) \times \Pi_{1\text{loop}}(k^2) \tag{1}$$

*Q.E.D.*

Problem 2(b):

Our next task is to calculate the  $\Pi_{1\text{loop}}(k^2)$  factor in eq. (1). As we saw in part (a), only the first diagram contributes to the correct tensor structure in  $\Sigma_{1\text{loop}}^{\mu\nu}$ , hence according to eq. (S.54)

$$\Pi_{1\text{loop}}(k^2) = e^2 \int_0^1 dx (1-2x)^2 \int_{\text{reg}} \frac{d^4 q}{(2\pi)^4} \frac{-i}{(q^2 - \Delta + i0)^2} \tag{S.63}$$

The momentum integral here should be rather familiar after so much related class- and home-work,

so let me simply write down the result:

$$\int_{\text{reg}} \frac{d^4 q}{(2\pi)^4} \frac{-i}{(q^2 - \Delta + i0)^2} = \frac{1}{16\pi^2} \Gamma\left(2 - \frac{D}{2}\right) \left(\frac{4\pi\mu^2}{\Delta}\right)^{2 - \frac{D}{2}} \xrightarrow{D \rightarrow 4} \frac{1}{16\pi^2} \left(\frac{1}{\bar{\epsilon}} + \log \frac{\mu^2}{\Delta}\right) \quad (\text{S.64})$$

where

$$\frac{1}{\bar{\epsilon}} = \frac{2}{4 - D} - \gamma_E + \log(4\pi). \quad (\text{S.65})$$

Consequently,

$$\begin{aligned} \Pi_{1\text{loop}}(k^2) &= \frac{\alpha}{4\pi} \int_0^1 dx (1 - 2x)^2 \left(\frac{1}{\bar{\epsilon}} + \log \frac{\mu^2}{M^2 - x(1-x)k^2}\right) \\ &= \frac{\alpha}{12\pi} \left(\frac{1}{\bar{\epsilon}} + \log \frac{\mu^2}{M^2} + \hat{I}(k^2/M^2)\right) \end{aligned} \quad (\text{S.66})$$

where

$$\hat{I}(k^2/M^2) = 3 \int_0^1 dx (1 - 2x)^2 \log \frac{M^2}{M^2 - x(1-x)k^2} = 3I(k^2/M^2) - 2J(k^2/M^2). \quad (\text{S.67})$$

Finally, note that eq. (S.66) is the bare one-loop amplitude, without accounting for the counterterms. Adding the counterterm  $\delta_3$  to the picture and imposing the renormalization condition (S.38), we arrive at

$$\delta_3^{1\text{loop}} = -\Pi_{1\text{loop}}(k^2 = 0) = -\frac{\alpha}{12\pi} \left(\frac{1}{\bar{\epsilon}} + \log \frac{\mu^2}{M^2}\right) \quad (\text{S.68})$$

while

$$\Pi_{\text{net}}(k^2) = +\frac{\alpha}{12\pi} \hat{I}(k^2/M^2) + O(\alpha^2) \quad (\text{S.69})$$

Problem 2(c):

At high momenta  $k^2 \gg M^2$ , we may approximate

$$\log \frac{M^2}{M^2 - x(1-x)k^2} \approx \log \frac{M^2}{-x(1-x)k^2} + O(M^2/k^2) \quad (\text{S.70})$$

and hence

$$\hat{I}(k^2/M^2) \approx 3 \int_0^1 dx (1-2x)^2 \left[ -\log \frac{-k^2}{M^2} + \log \frac{1}{x(1-x)} \right] = -\log \frac{-k^2}{M^2} + \frac{8}{3}. \quad (\text{S.71})$$

Consequently, at high momenta the “vacuum polarization” factor  $\Pi(k^2)$  behaves as

$$\Pi(k^2) = \frac{\alpha}{12\pi} \left( -\log \frac{-k^2}{M^2} + \frac{8}{3} + O(M^2/k^2) \right) + O(\alpha^2), \quad (\text{S.72})$$

and therefore the effective gauge coupling

$$\alpha_{\text{eff}}(k^2) = \frac{\alpha}{1 + \Pi(k^2)} \quad (\text{S.73})$$

behaves according to

$$\frac{1}{\alpha_{\text{eff}}(k^2)} \approx \frac{1}{\alpha(0)} - \frac{1}{12\pi} \left( \log \frac{-k^2}{M^2} - \frac{8}{3} \right). \quad (2)$$

*Q.E.D.*