Mechanics - Basic Physical Concepts

Mathematics

Quadratic Eq.: $a\,x^2 + b\,x + c = 0,\ x = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$

Cartesian and polar coordinates:

$x = r\cos\theta,\ y = r\sin\theta, \quad r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$

Trigonometry: $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

$\sin\alpha + \sin\beta = 2\,\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha + \cos\beta = 2\,\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\sin 2\,\theta = 2\,\sin\theta\cos\theta, \cos 2\,\theta = \cos^2\,\theta - \sin^2\,\theta$

$1 - \cos\theta = 2\,\sin^2\frac{\theta}{2}, \quad 1 + \cos\theta = 2\,\cos^2\frac{\theta}{2}$

Vector algebra: $\vec{A} = (A_x, A_y) = A_x\hat{\imath} + A_y\hat{\jmath}$

Resultant: $\vec{R} = \vec{A} + \vec{B} = (A_x + B_x, A_y + B_y)$

Dot: $\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_x\,B_x + A_y\,B_y + A_z\,B_z$

Cross product: $\hat{\imath}\times\hat{\jmath} = \hat{k}, \quad \hat{\jmath}\times\hat{k} = \hat{\imath}, \quad \hat{k}\times\hat{\imath} = \hat{\jmath}$

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$C = A\,B\sin\theta = A_\perp B = A\,B_\perp,$ use right hand rule

Calculus: $\frac{d}{dx}\,x^n = n\,x^{n-1}, \quad \frac{d}{dx}\ln x = \frac{1}{x},$

$\frac{d}{d\theta}\sin\theta = \cos\theta, \quad \frac{d}{d\theta}\cos\theta = -\sin\theta, \quad \frac{d}{dx}\,\text{const} = 0$

Measurements

Dimensional analysis: *e.g.*,

$F = m\,a \to [M][L][T]^{-2}, \quad$ or $F = m\,\frac{v^2}{R} \to [M][L][T]^{-2}$

Summation: $\sum_{i=1}^{N}(a\,x_i + b) = a\sum_{i=1}^{N} x_i + b\,N$

Motion

One dimensional motion: $v = \frac{d\,s}{dt}, \quad a = \frac{d\,v}{dt}$

Average values: $\bar{v} = \frac{s_f - s_i}{t_f - t_i}, \quad \bar{a} = \frac{v_f - v_i}{t_f - t_i}$

One dimensional motion (constant acceleration):

$v(t): \quad v = v_0 + a\,t$

$s(t): \quad s = \bar{v}\,t = v_0\,t + \frac{1}{2}\,a\,t^2, \quad \bar{v} = \frac{v_0 + v}{2}$

$v(s): \quad v^2 = v_0^2 + 2\,a\,s$

Nonuniform acceleration: $x = x_0 + v_0\,t + \frac{1}{2}\,a\,t^2 + \frac{1}{6}\,j\,t^3 + \frac{1}{24}\,s\,t^4 + \frac{1}{120}\,k\,t^5 + \frac{1}{720}\,p\,t^6 + \ldots$, (jerk, snap,...)

Projectile motion: $t_{rise} = t_{fall} = \frac{t_{trip}}{2} = \frac{v_{0y}}{g}$

$h = \frac{1}{2}\,g\,t_{fall}^2, \ R = v_{ox}\,t_{trip}$

Circular: $a_c = \frac{v^2}{r}, \quad v = \frac{2\,\pi\,r}{T}, \quad f = \frac{1}{T}$ (Hertz=s^{-1})

Curvilinear motion: $a = \sqrt{a_t^2 + a_r^2}$

Relative velocity: $\vec{v} = \vec{v}\,' + \vec{u}$

Law of Motion and applications

Force: $\vec{F} = m\,\vec{a},\ F_g = m\,g,\ \vec{F}_{12} = -\vec{F}_{21}$

Circular motion: $a_c = \frac{v^2}{r},\ v = \frac{2\,\pi\,r}{T} = 2\,\pi\,r\,f$

Friction: $F_{static} \le \mu_s\,N \qquad F_{kinetic} = \mu_k\,N$

Equilibrium (concurrent forces): $\sum_i \vec{F}_i = 0$

Energy

Work (for all F): $\Delta W = W_{A\to B} = W_B - W_A = F\,s_\parallel = F_\parallel s = F s\cos\theta = \vec{F}\cdot\vec{s} \to \int_A^B \vec{F}\cdot d\vec{s}$ (in Joules)

Effects due to work done: $F_{ext} = m\,a + F_c + f_{nc}$

$W_{ext}|_{A\to B} = K_B - K_A + U_B - U_A + W_{diss}|_{A\to B}$

Kinetic energy: $K_B - K_A = \int_A^B m\,\vec{a}\cdot d\vec{s}, \quad K = \frac{1}{2}\,m\,v^2$

K (conservative $\vec{F}$): $U_B - U_A = -\int_A^B \vec{F}\cdot d\vec{s}$

$U_{gravity} = m\,g\,y, \qquad U_{spring} = \frac{1}{2}\,k\,x^2$

From U to $\vec{F}$: $F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$

$F_{gravity} = -\frac{\partial U}{\partial y} = -m\,g, \quad F_{spring} = -\frac{\partial U}{\partial x} = -k\,x$

Equilibrium: $\frac{\partial U}{\partial x} = 0, \quad \frac{\partial^2 U}{\partial x^2} > 0$ stable, < 0 unstable

Power: $P = \frac{d\,W}{dt} = F\,v_\parallel = F\,v\cos\theta = \vec{F}\cdot\vec{v}$ (Watts)

Collision

Impulse: $\vec{I} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i \to \int_{t_i}^{t_f}\vec{F}\,dt$

Momentum: $\vec{p} = m\,\vec{v}$

Two-body: $x_{cm} = \frac{m_1\,x_1 + m_2\,x_2}{m_1 + m_2}$

$p_{cm} \equiv M\,v_{cm} = p_1 + p_2 = m_1\,v_1 + m_2\,v_2$

$F_{cm} \equiv F_1 + F_2 = m_1\,a_1 + m_2\,a_2 = M\,a_{cm}$

$K_1 + K_2 = K_1^* + K_2^* + K_{cm}$

Two-body collision: $\vec{p}_i = \vec{p}_f = (m_1 + m_2)\,\vec{v}_{cm}$

$v_i^* = v_i - v_{cm}, \qquad v_i' = v_i^{*\prime} + v_{cm}$

Elastic: $v_1 - v_2 = -(v_1' - v_2'),$

$v_i^{*\prime} = -v_i^*, \quad v_i' = 2\,v_{cm} - v_i$

Many body center of mass: $\vec{r}_{cm} = \frac{\sum m_i\,\vec{r}_i}{\sum m_i} = \frac{\int \vec{r}\,dm}{\int m_i}$

Force on cm: $\vec{F}_{ext} = \frac{d\vec{p}}{dt} = M\vec{a}_{cm}, \quad \vec{p} = \sum\vec{p}_i$

Rotation of Rigid-Body

Kinematics: $\theta = \frac{s}{r}, \quad \omega = \frac{v}{r}, \quad \alpha = \frac{a_t}{r}$

Moment of inertia: $I = \sum m_i\,r_i^2 = \int r^2 dm$

$I_{disk} = \frac{1}{2}\,M\,R^2, \quad I_{ring} = \frac{1}{2}\,M\,(R_1^2 + R_2^2)$

$I_{rod} = \frac{1}{12}\,M\,\ell^2, \quad I_{rectangle} = \frac{1}{12}\,M\,(a^2 + b^2)$

$I_{sphere} = \frac{2}{5}\,M\,R^2, \quad I_{spherical\ shell} = \frac{2}{3}\,M\,R^2$

$I = M\,(\text{Radius of gyration})^2, \quad I = I_{cm} + M\,D^2$

Kinetic energies: $K_{rot} = \frac{1}{2}\,I\,\omega^2, \quad K = K_{rot} + K_{cm}$

Angular momentum: $L = r\,m\,v = r\,m\,\omega\,r = I\,\omega$

Torque: $\tau = \frac{d\,L}{dt} = m\,\frac{d\,v}{dt}\,r = F\,r = I\,\frac{d\omega}{dt} = I\,\alpha$

$W_{ext} = \Delta K + \Delta U + W_f, \quad K = K_{rot} + \frac{1}{2}\,m\,v^2, \quad P = \tau\,\omega$

Rolling, angular momentum and torque

Rolling: $K = \frac{1}{2}\left(I_c + M\,R^2\right)\omega^2 = \frac{1}{2}\left(\frac{I_c}{R^2} + M\right)v^2$

Angular momentum: $\vec{L} = \vec{r}\times\vec{p},\ L = r_\perp\,p = I\,\omega$

Torque: $\vec{\tau} = \frac{d\,\vec{L}}{dt} = \vec{r}\times\frac{d\,\vec{p}}{dt} = \vec{r}\times\vec{F}, \quad \tau = r_\perp\,F = I\,\alpha$

Gyroscope: $\omega_p = \frac{d\,\phi}{dt} = \frac{1}{L}\frac{d\,L}{dt} = \frac{\tau}{L} = \frac{m\,g\,h}{I\,\omega}$

Static equilibrium

$\sum\vec{F}_i = 0, \quad$ about any point $\sum\vec{\tau}_i = 0$

Subdivisions: $\vec{r}_{cm} = \frac{m_A\,\vec{r}_{Acm} + m_B\,\vec{r}_{Bcm}}{m_A + m_B}$

Elastic modulus = stress/strain

stress: F/A

strain: $\Delta L/L, \quad \theta \approx \Delta x/h, \quad -\Delta V/V$

Gravity

$$\vec{F}_{21} = -G\,\frac{m_1 m_2}{r_{12}^2}\,\hat{r}_{12}, \quad \text{for } r \geq R, \quad g(r) = G\,\frac{M}{r^2}$$

$$G = 6.67259 \times 10^{-11}\ \mathrm{N\,m^2/kg^2}$$

$$R_{earth} = 6370\ \text{km}, \quad M_{earth} = 5.98 \times 10^{24}\ \text{kg}$$

Circular orbit: $a_c = \frac{v^2}{r} = \omega^2\,r = \left(\frac{2\pi}{T}\right)^2 r = g(r)$

$$U = -G\,\frac{m\,M}{r}, \quad E = U + K = -\frac{G\,m\,M}{2\,r}$$

$$F = -\frac{dU}{dr} = -m\,G\,\frac{M}{r^2} = -m\,\frac{v^2}{r}$$

Kepler's Laws of planetary motion:

i) elliptical orbit, $r = \frac{r_0}{1-\epsilon\cos\theta}$ $\quad r_1 = \frac{r_0}{1+\epsilon}, \quad r_2 = \frac{r_0}{1-\epsilon}$

ii) $L = r\,m\,\frac{\Delta r_\perp}{\Delta t} \longrightarrow \frac{\Delta A}{\Delta t} = \frac{1}{2}\,\frac{r\,\Delta r_\perp}{\Delta t} = \frac{L}{2\,m} = \text{const.}$

iii) $G\,\frac{M}{a^2} = \left(\frac{2\,\pi\,a}{T}\right)^2 \frac{1}{a}, \quad a = \frac{r_1+r_2}{2}, \quad T^2 = \left(\frac{4\,\pi^2}{G\,M}\right) r^3$

Escape kinetic energy: $E = K + U(R) = 0$

Fluid mechanics

Pascal: $P = \frac{F_{\perp 1}}{A_1} = \frac{F_{\perp 2}}{A_2}, \quad 1\ \text{atm} = 1.013 \times 10^5\ \mathrm{N/m^2}$

Archimedes: $B = M\,g, \quad \text{Pascal}=\mathrm{N/m^2}$

$P = P_{atm} + \rho\,g\,h, \quad$ with $P = \frac{F_\perp}{A}$ and $\rho = \frac{m}{V}$

$F = \int P\,dA \longrightarrow \rho\,g\,\ell\int_0^h (h-y)\,dy$

Continuity equation: $A\,v = \text{constant}$

Bernoulli: $P + \frac{1}{2}\,\rho\,v^2 + \rho\,g\,y = \text{const}, \qquad P \geq 0$

Oscillation motion

$$f = \frac{1}{T}, \quad \omega = \frac{2\,\pi}{T}$$

S H M: $a = \frac{d^2\,x}{dt^2} = -\omega^2\,x, \quad \alpha = \frac{d^2\,\theta}{dt^2} = -\omega^2\,\theta$

$$x = x_{max}\ \cos(\omega\,t + \delta), \quad x_{max} = A$$

$$v = -v_{max}\ \sin(\omega\,t + \delta), \quad v_{max} = \omega\,A$$

$$a = -a_{max}\ \cos(\omega\,t + \delta) = -\omega^2\,x, \quad a_{max} = \omega^2\,A$$

$$E = K + U = K_{max} = \frac{1}{2}\,m\,(\omega\,A)^2 = U_{max} = \frac{1}{2}\,k\,A^2$$

Spring: $m\,a = -k\,x$

Simple pendulum: $m\,a_\theta = m\,\alpha\,\ell = -m\,g\,\sin\theta$

Physical pendulum: $\tau = I\,\alpha = -m\,g\,d\sin\theta$

Torsion pendulum: $\tau = I\,\alpha = -\kappa\,\theta$

Wave motion

Traveling waves: $y = f(x - v\,t), \quad y = f(x + v\,t)$

In the positive x direction: $y = A\,\sin(k\,x - \omega\,t - \phi)$

$$T = \frac{1}{f}, \quad \omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}, \quad v = \frac{\omega}{k} = \frac{\lambda}{T}$$

Along a string: $v = \sqrt{\frac{F}{\mu}}$

Reflection of wave: fixed end: *phase inversion*; open end: *same phase*

General: $\Delta E = \Delta K + \Delta U = \Delta K_{max}$

$$P = \frac{\Delta E}{\Delta t} = \frac{1}{2}\,\frac{\Delta m}{\Delta t}\,(\omega A)^2$$

Waves: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta x} \cdot \frac{\Delta x}{\Delta t} = \frac{\Delta m}{\Delta x} \cdot v$

$$P = \frac{1}{2}\,\mu\,v\,(\omega\,A)^2, \quad \text{with } \mu = \frac{\Delta m}{\Delta x}$$

Circular: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta A} \cdot \frac{\Delta A}{\Delta r} \cdot \frac{\Delta r}{dt} = \frac{\Delta m}{\Delta A} \cdot 2\,\pi\,r\,v$

Spherical: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta V} \cdot 4\,\pi\,r^2\,v$

Sound

$$v = \sqrt{\frac{B}{\rho}}, \quad s = s_{max}\cos(k\,x - \omega\,t - \phi)$$

$$\Delta P = -B\,\frac{\Delta V}{V} = -B\,\frac{\partial s}{\partial x}$$

$$\Delta P_{max} = B\,\kappa\,s_{max} = \rho\,v\,\omega\,s_{max}$$

Piston: $\frac{\Delta m}{\Delta t} = \frac{\Delta m}{\Delta V} \cdot \frac{A\,\Delta x}{\Delta t} = \rho\,A\,v$

Intensity: $I = \frac{P}{A} = \frac{1}{2}\,\rho\,v\,(\omega\,s_{max})^2$

Intensity level: $\beta = 10\log_{10}\frac{I}{I_0}, \ \ I_0 = 10^{-12}\ \mathrm{W/m^2}$

Plane waves: $\psi(x,t) = c\,\sin(k\,x - \omega\,t)$

Circular waves: $\psi(r,t) = \frac{c}{\sqrt{r}}\sin(k\,r - \omega\,t)$

Spherical: $\psi(r,t) = \frac{c}{r}\sin(k\,r - \omega\,t)$

Doppler effect: $\lambda = v\,T, \quad f_0 = \frac{1}{T}, \quad f' = \frac{v'}{\lambda'}$

Here $v' = v_{sound} \pm v_{observer}$, is wave speed relative to moving observer and $\lambda' = (v_{sound} \pm v_{source})/f_0$, detected wave length established by moving source of frequency f_0. $\ f_{received} = f_{reflected}$

Shock waves: Mach Number$= \frac{v_{source}}{v_{sound}} = \frac{1}{\sin\theta}$

Superposition of waves

Phase difference: $\sin(k\,x - \omega t) + \sin(k\,x - \omega\,t - \phi)$

Standing waves: $\sin(k\,x - \omega\,t) + \sin(k\,x + \omega\,t)$

Beats: $\sin(kx - \omega_1\,t) + \sin(k\,x - \omega_2\,t)$

Fundamental modes: Sketch wave patterns

String: $\frac{\lambda}{2} = \ell$, *Rod clamped middle:* $\frac{\lambda}{2} = \ell$,

Open-open pipe: $\frac{\lambda}{2} = \ell$, *Open-closed pipe:* $\frac{\lambda}{4} = \ell$

Temperature and heat

Conversions: $F = \frac{9}{5}\,C + 32°, \qquad K = C + 273.15°$

Constant volume gas thermometer: $T = a\,P + b$

Thermal expansion: $\alpha = \frac{1}{\ell}\,\frac{d\,\ell}{dT}, \quad \beta = \frac{1}{V}\,\frac{d\,V}{dT}$

$$\Delta\ell = \alpha\,\ell\,\Delta T, \quad \Delta A = 2\,\alpha\,A\,\Delta T, \quad \Delta V = 3\,\alpha\,V\,\Delta T$$

Ideal gas law: $\qquad P\,V = n\,R\,T = N\,k\,T$

$$R = 8.314510\ \text{J/mol/K} = 0.0821\ \text{L atm/mol/K}$$

$k = 1.38 \times 10^{-23}$J/K, $\ N_A = 6.02 \times 10^{23}$, 1 cal=4.19 J

Calorimetry: $\Delta Q = c\,m\,\Delta T, \ \Delta Q = L\,\Delta m$

First law: $\Delta U = \Delta Q - \Delta W, \quad W = \int P\,dV$

Conduction: $H = \frac{\Delta Q}{\Delta t} = -k\,A\,\frac{\Delta T}{\Delta\ell}, \quad \Delta T_i = \frac{-H}{A}\,\frac{\ell_i}{k_i}$

Stefan's law: $P = \sigma\,A\,e\,T^4, \quad \sigma = 5.67 \times 10^{-8}\,\frac{W}{m^2\,K^4}$

Kinetic theory of gas

Ideal gas: $\Delta p_x = 2\,m\,v_x, \quad F = \frac{\Delta p_x}{\Delta t} = \frac{m\,v_x^2}{d}$

Pressure: $P = \frac{N\,\overline{F}}{A} = \frac{m\,N}{V}\,\overline{v_x^2} = \frac{m\,N}{3\,V}\,\overline{v^2}$

$$P = \frac{2}{3}\,\frac{N}{V}\,\overline{K}, \quad \overline{K}_x = \frac{\overline{K}}{3} = \frac{1}{2}\,k\,T, \quad T = 273 + T_c,$$

$$P\,V = N\,k\,T, \quad n = N/N_A, \quad k = 1.38{\times}10^{-23}\ \text{J/K},$$

$$N_A = 6.02214199 \times 10^{23}\ \text{\#/kg/mole}$$

Constant V: $\qquad \Delta Q = \Delta U = n\,C_V\,\Delta T$

Constant P: $\qquad \Delta Q = n\,C_P\,\Delta T$

$$\gamma = \frac{C_P}{C_V}, \quad C_P - C_V = R$$

$$C_V = \frac{d}{2}\,R, \quad \text{for transl.+rot+vib}, \quad d = 3 + 2 + 2$$

Adiabatic expansion: $P\,V^\gamma = \text{constant}$

Mean free path: $\ell = \frac{v_{rms}\,t}{(v_{rel})_{rms}\,t}\,\frac{1}{\pi\,d^2\,n_V} = \frac{1}{\sqrt{2}\,\pi\,d^2\,n_V}$