Textbook problem **5.5**:

Strictly speaking, the problem of gravity on a 200 kilometer (125 mile) mountain is way too difficult for this class because a planet with a mountain *that big* would have a different gravitational field than the almost-spherical Earth we live on. So let's rephrase the question and assume a 200 km *ladder* instead of a mountain, and also assume that the ladder's own mass is too small to create a noticeable gravity of its own. Or more realistically, assume a stratospheric balloon floating 200 km up in the (very thin) air. In any case, assume the Earth to be a perfect sphere and consider your weight 200 km above the ground.

In general, gravitational fields near large bodies are quite complicated, but for a spherically symmetric body the field outside the body is simply the same as if the entire mass was in the body's center. Thus, the gravitational field outside the Earth is

$$g(r) \;=\; \frac{GM_{\text{Earth}}}{r^2} \tag{1}$$

where r is the distance from the Earth's center. At altitude h above the ground $r = h + R_{\text{Earth}}$, hence g changes with the altitude according to

$$g(h) \;=\; \frac{GM_E}{(R_E + h)^2}\,. \tag{2}$$

Consequently, your weight $W = mg$ depends on your altitude according to

$$W(h) \;=\; m \times g(h) \;=\; m \times \frac{GM_E}{(R_E + h)^2} \tag{3}$$

where m is your mass.

Let us compare this weight with your weight on the surface

$$W(0) \;=\; m \times g(0) \tag{4}$$

where

$$g(0) \;=\; \frac{GM_E}{R_E^2} \;\approx\; 9.8 \text{ m/s}^2. \tag{5}$$

Dividing eq. (3) by eq. (4) we have

$$\frac{W(h)}{W(0)} \;=\; \frac{g(h)}{g(0)} \;=\; \frac{GM_E}{(R_E+h)^2} \Bigg/ \frac{GM_E}{R_E^2} \;=\; \left(\frac{R_E}{R_E+h}\right)^2. \tag{6}$$

In particular, at altitude $h = 200$ km

$$\frac{W(h)}{W(0)} \;=\; \left(\frac{6378\,\text{km}}{6378\,\text{km} \;+\; 200\,\text{km}}\right)^2 \;\approx\; 0.94 \tag{7}$$

which means that your weight is 94% of your weight on the surface. In other words, when you go 200 km up, your weight decreases by 6%.

For example, if you normally weigh 150 pounds, then 200 km up you would weigh 0.94×150 lb $= 141$ lb, 9 pounds less than on the surface. This is a noticeable difference, easily detectable by ordinary bathroom scales.

Note that the problem assumes you are *standing in place* at altitude $h = 200$ km rather than *orbiting* up there in a space shuttle or other kind of spaceship. In an orbiting spaceship, you would be in free fall and the inertial force due to spaceship's centripetal acceleration would precisely cancel your weight, hence you would feel completely weightless. On the other hand, if you are standing up there on a ladder or in a balloon's gondola, there would be no inertial force and you you would feel the gravitational field as it is at $h = 200$ km. This gravity is a little weaker than on the surface, but only a little, so you would feel 94% of your normal weight.

Non-textbook problem **1**:

Titan is fairly spherical in shape, so your weight on the surface of Titan is given by

$$W(\text{onTitan}) \;=\; m \times g_T \;=\; m \times \frac{GM_T}{R_T^2} \tag{8}$$

where m is your mass, g_T is Titan's gravity, M_T is Titan's mass, and R_T is Titan's radius. Comparing this to your weight on Earth given by eqs. (4) and (5), we have

$$\begin{aligned}\frac{W(\text{onTitan})}{W(\text{onEarth})} &= \frac{g_T}{g_E}\\ &= \frac{GM_T}{R_T^2} \Bigg/ \frac{GM_E}{R_E^2}\\ &= \frac{M_T}{M_E} \Bigg/ \left(\frac{R_T}{R_E}\right)^2\\ &= 0.0226/0.468^2 \;=\; 0.103.\end{aligned} \tag{9}$$

In other words, you weight on Titan is only 10.3% of your weight on Earth. For example, if you weigh 150 pounds on Earth, you weight on Titan would be only 15.5 pounds.

Non-textbook problem **2**:

The year on another planet is the time the planet takes to make one orbit around its star. For a circular orbit of radius R, we can calculate this time from the orbit equation for the planet's speed v,

$$\frac{GM_\star}{R^2} \;=\; \frac{v^2}{R} \;\Longrightarrow\; R \times v^2 \;=\; GM_\star \tag{10}$$

where $M_\star$ is the star's mass. Using $v = 2\pi R/T$ where T is the orbital period, we find

$$R^3 \times \left(\frac{2\pi}{T}\right)^2 \;=\; GM_\star \tag{11}$$

and hence the year is

$$T = 2\pi \times \sqrt{\frac{R^3}{GM_\star}}\,. \tag{12}$$

For elliptical orbits, one needs calculus to calculate the orbital period, but the result is fairly simple:

$$T = 2\pi \times \sqrt{\frac{a^3}{GM_\star}} \tag{13}$$

where a is the semi-major axis of the planetary orbit. For a circular orbit, eq. (13) reduces to eq. (12).

Now let's compare eq. (13) for a planet orbiting another star to a similar equation

$$T_E = 2\pi \times \sqrt{\frac{a_E^3}{GM_\odot}} \tag{14}$$

for the Earth orbiting the Sun; here $a_E = 1$ au is the radius (or rather semi-major axis) of the Earth's orbit and $M_\odot$ is the Sun's mass. Dividing eq. (13) by eq. (14) we obtain

$$\frac{T}{T_E} = \sqrt{\frac{a^3}{GM_\star}} \Bigg/ \sqrt{\frac{a_E^3}{GM_\odot}} = \sqrt{\left(\frac{a}{a_E}\right)^3 \Bigg/ \left(\frac{M_\star}{M_\odot}\right)}\,, \tag{15}$$

hence for the orbit of radius $a = 2$ au $\equiv 2 \times a_E$ around a star of mass $M_\star = 3 \times M_\odot$ we have

$$\frac{T}{T_E} = \sqrt{2^3/3} = \sqrt{8/3} = 1.633. \tag{16}$$

In other words, the planet's year is 1.633 Earth years, or 596.5 Earth days, or 5.153×10^7 seconds.

Non-textbook problem **3**:

A satellite of Mars with a circular orbit of radius R has period

$$T \;=\; 2\pi\times\sqrt{\frac{R^3}{GM_{\rm Mars}}}\,, \tag{17}$$

(*cf.* eq. (12)). An "arestationary" satellite has its period equal to 1 martian day+night, thus $T=T_{MD}=88775$ s, hence

$$2\pi\times\sqrt{\frac{R^3}{GM_{\rm Mars}}} \;=\; T_{MD}\,. \tag{18}$$

Solving this equation for the orbit's radius, we have

$$\frac{R^3}{GM_{\rm Mars}} \;=\; \left(\frac{T_{MD}}{2\pi}\right)^2 \tag{19}$$

and consequently

$$\begin{aligned} R \;&=\; \sqrt[3]{GM_{\rm Mars}\times(T_{MD}/2\pi)^2}\\ &=\; \sqrt[3]{(6.67\times10^{-11}\,m^3/s^2/kg)\times(6.42\times10^{23}\,kg)\times(88775\,s/2\pi)^2}\\ &=\; 2.045\times10^7\ \text{m} \;=\; 20450\text{km} \;\approx\; 12700\ \text{miles}. \end{aligned} \tag{20}$$