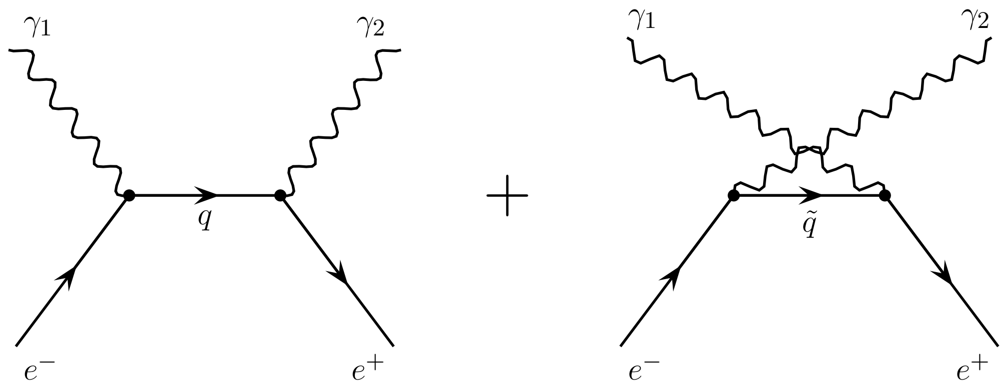
ANNIHILATION

In these notes I explain the $e^+e^- \to \gamma\gamma$ annihilation process. At the tree level of QED, there are two diagrams related by interchanging of the two photons in the final state:



(1)

The net amplitude due to these diagrams is

$$\begin{aligned}
\mathcal{M} &= e^*_\mu(k_1,\lambda_1)\, e^*_\nu(k_2,\lambda_2) \times \mathcal{M}^{\mu\nu}, \\
\mathcal{M}^{\mu\nu} &= \mathcal{M}_1^{\mu\nu} \;+\; \mathcal{M}_2^{\mu\nu}, \\
i\mathcal{M}_1^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\nu)\frac{i}{\not{q}-m}(ie\gamma^\mu)u(e^-), \\
i\mathcal{M}_2^{\mu\nu} &= \bar{v}(e^+)(ie\gamma^\mu)\frac{i}{\not{\tilde{q}}-m}(ie\gamma^\nu)u(e^-),
\end{aligned} \tag{2}$$

where $q = p_- - k_1 = k_2 - p_+$ and $\tilde{q} = p_- - k_2 = k_1 - p_+$. Note the opposite orders of the γ^μ and γ^ν vertices in the last two lines. We may use

$$\frac{1}{\not{q}-m} = \frac{\not{q}+m}{q^2-m^2} = \frac{\not{q}+m}{t-m^2} \quad \text{and} \quad \frac{1}{\not{\tilde{q}}-m} = \frac{\not{\tilde{q}}+m}{q^2-m^2} = \frac{\not{\tilde{q}}+m}{u-m^2} \tag{3}$$

to avoid matrix denominators in the amplitudes: The last two lines of eq. (2) become

$$\begin{aligned}
\mathcal{M}_1^{\mu\nu} &= \frac{-e^2}{t-m^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u, \\
\mathcal{M}_2^{\mu\nu} &= \frac{-e^2}{u-m^2} \times \bar{v}\gamma^\mu(\not{\tilde{q}}+m)\gamma^\nu u.
\end{aligned} \tag{4}$$

Ward Identity

Before we go any further, lets check the Ward identities for the annihilation amplitude: for the first photon we should have $k_{1\mu}\mathcal{M}^{\mu\nu} = 0$, and for the second photon $k_{2\nu}\mathcal{M}^{\mu\nu} = 0$. Let's start with the first photon and the first diagram. Multiplying the second factor in the first eq. (4) by $k_{1\mu}$, we have

$$\begin{aligned}
\bar{v}\gamma^\nu(\not{q} + m)\gamma^\mu u \times k_{1\mu} &= \bar{v}\gamma^\nu(\not{p}_- - \not{k}_1 + m)\not{k}_1 u \\
&= \bar{v}\gamma^\nu(\not{p}_- + m)\not{k}_1 u \quad \langle\!\langle \text{because } \not{k}_1\not{k}_1 = k_1^2 = 0 \rangle\!\rangle \\
&= \bar{v}\gamma^\nu\Big(2(p_-k_1) - \not{k}_1(\not{p}_- - m)\Big)u \\
&= 2(p_-k_1) \times \bar{v}\gamma^\nu u \quad \langle\!\langle \text{because } (\not{p}_- - m)u = 0 \rangle\!\rangle \\
&= (m^2 - t) \times \bar{v}\gamma^\nu u
\end{aligned} \tag{5}$$

and consequently

$$\mathcal{M}_1^{\mu\nu} \times k_{1\mu} = +e^2 \times \bar{v}\gamma^\nu u\,. \tag{6}$$

Note the non-zero right hand side — the first diagram does not satisfy the Ward identity all by itself. As for the second diagram, we have

$$\begin{aligned}
\bar{v}\gamma^\mu(\tilde{\not{q}} + m)\gamma^\nu u \times k_{1\mu} &= \bar{v}\not{k}_1(\not{k}_1 - \not{p}_+ + m)\gamma^\nu u \\
&= \bar{v}\not{k}_1(-\not{p}_+ + m)\gamma^\nu u \quad \langle\!\langle \text{because } \not{k}_1\not{k}_1 = k_1^2 = 0 \rangle\!\rangle \\
&= \bar{v}\Big(-2(p_+k_1) + (\not{p}_+ + m)\not{k}_1\Big)\gamma^\nu u \\
&= -2(p_+k_1) \times \bar{v}\gamma^\nu u \quad \langle\!\langle \text{because } \bar{v}(\not{p}_+ + m) = 0 \rangle\!\rangle \\
&= -(m^2 - u) \times \bar{v}\gamma^\nu u
\end{aligned} \tag{7}$$

and consequently

$$\mathcal{M}_1^{\mu\nu} \times k_{1\mu} = -e^2 \times \bar{v}\gamma^\nu u\,. \tag{8}$$

Again we have a non-zero result — the second diagram also does not satisfy the Ward identity all by itself. However, the right hand sides of eqs. (6) and (8) cancel each other, so *together*, the two diagrams do satisfy the Ward identity:

$$\mathcal{M}^{\mu\nu} \times k_{1\mu} \;=\; 0. \tag{9}$$

This is an example of a general rule: The Ward Identity does not work diagram by diagram but only for entire amplitudes, or for partial sums of all diagrams related by permutations of photonic vertices on the same fermionic line.

The Ward identity $\mathcal{M}^{\mu\nu} \times k_{2\nu} = 0$ for the second photon works similarly to the first, and I see no point in repeating the argument. Indeed, it would be an exactly similar argument because the net annihilation amplitude is symmetric with respect to the two photons.

Summing over the Spins and Polarizations

Earlier in class I explained how to use Ward identities to sum $|\mathcal{M}|^2$ over polarizations of the two photons:

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; +\mathcal{M}^{\mu\nu}\mathcal{M}^*_{\mu\nu}\,. \tag{10}$$

Combining the two diagrams, we have

$$\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\; \mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{1\mu\nu} \;+\; \mathcal{M}_2^{\mu\nu}\mathcal{M}^*_{2\mu\nu} \;+\; 2\Re\mathcal{M}_1^{\mu\nu}\mathcal{M}^*_{2\mu\nu}\,. \tag{11}$$

Note that this formula works despite the fact that $\mathcal{M}_1^{\mu\nu}$ and $\mathcal{M}_2^{\mu\nu}$ do not satisfy the Ward Identities by themselves — it's enough that the sum $\mathcal{M}_1^{\mu\nu} + \mathcal{M}_2^{\mu\nu}$ satisfies the identities. Thus, in light of eqs. (4),

$$\begin{aligned}\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \;=\;& \frac{e^4}{(t-m^2)^2} \times \bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\mu(\not{q}+m)\gamma_\nu v \\ &+\; \frac{e^4}{(u-m^2)^2} \times \bar{v}\gamma^\mu(\tilde{\not{q}}+m)\gamma^\nu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v \\ &+\; \frac{2e^4}{(t-m^2)(u-m^2)} \times \Re\Big(\bar{v}\gamma^\nu(\not{q}+m)\gamma^\mu u \times \bar{u}\gamma_\nu(\tilde{\not{q}}+m)\gamma_\mu v\Big).\end{aligned} \tag{12}$$

This takes care of the photon polarizations. The next step is to average over sins of the initial electron and positron. Proceeding is usual, we have

$$\begin{aligned}\overline{|\mathcal{M}|^2} \;&\equiv\; \tfrac14 \sum_{s_-,s_+}\sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 \\ &=\; \frac{e^4}{(t-m^2)^2}\times A_{11} \;+\; \frac{e^4}{(u-m^2)^2}\times A_{22} \;+\; \frac{2e^4}{(t-m^2)(u-m^2)}\times \Re\, A_{12}\,,\end{aligned} \tag{13}$$

where

$$\begin{aligned} A_{11} &= \tfrac14 \operatorname{Tr}\Big((\not p_+ - m)\gamma^\nu(\not q + m)\gamma^\mu(\not p_- + m)\gamma_\mu(\not q + m)\gamma_\nu \Big),\\ A_{22} &= \tfrac14 \operatorname{Tr}\Big((\not p_+ - m)\gamma^\mu(\tilde{\not q} + m)\gamma^\nu(\not p_- + m)\gamma_\nu(\tilde{\not q} + m)\gamma_\mu \Big),\\ A_{12} &= \tfrac14 \operatorname{Tr}\Big((\not p_+ - m)\gamma^\nu(\not q + m)\gamma^\mu(\not p_- + m)\gamma_\nu(\tilde{\not q} + m)\gamma_\mu \Big),\end{aligned} \tag{14}$$

Traceology 1

Our next task is to evaluate the traces (14). Let's start with the A_{11}.

Back in homework set #6 (problem 1.d), you saw that $\gamma^\mu\gamma_\mu = 4$ and $\gamma^\mu \not p \gamma_\mu = -2\not p$. Applying these formulae to the expression inside the trace in A_{11}, we have

$$\gamma^\mu(\not p_- + m)\gamma_\mu \;=\; -2(\not p_- - 2m), \qquad \gamma_\nu(\not p_+ - m)\gamma^\nu \;=\; -2(\not p_+ + 2m), \tag{15}$$

and consequently

$$A_{11} \;=\; \operatorname{Tr}\Big((\not p_+ + 2m)(\not q + m)(\not p_- - 2m)(\not q + m) \Big). \tag{16}$$

Next, we expand the parentheses inside this trace and throw away terms with odd numbers of momenta $\not p$ or $\not q$. This gives us

$$\begin{aligned} A_{11} \;&=\; \operatorname{Tr}(\not p_+ \not q \not p_- \not q) \;+\; m^2 \operatorname{Tr}(\not p_+ \not p_-) \;-\; 4m^2 \operatorname{Tr}(\not q \not q) \\ &\qquad +\; 2\times 2m^2 \operatorname{Tr}(\not p_- \not q) \;-2\times 2m^2 \operatorname{Tr}(\not p_+ \not q) \;-\; 4m^4 \operatorname{Tr}(1) \\ &=\; 8(p_+q)(p_-q) \;-\; 4(p_+p_-)\, q^2 \\ &\qquad +\; 4m^2(p_+p_-) \;-\; 16m^2q^2 \;+\; 16m^2(p_+q) \;-\; 16m^2(p_-q) \;-\; 16m^4\,. \end{aligned} \tag{17}$$

Finally, let's express all the kinematic quantities in terms of the Mandelstam's variables

s, t, and u. Using $p_-^2 = p_+^2 = m^2$ and $k_1^2 = k_2^2 = 0$, we have

$$\begin{aligned} q^2 &= (p_- - k_1)^2 = t, \\ qp_- &= (p_- - k_1)p_- = m^2 - p_-k_1 = m^2 + \tfrac{1}{2}(t - m^2) = +\tfrac{1}{2}(m^2 + t), \\ qp_+ &= (k_2 - p_+)p_+ = p_+k_2 - m^2 = -\tfrac{1}{2}(t - m^2) - m^2 = -\tfrac{1}{2}(t + m^2), \\ p_-p_+ &= \tfrac{1}{2}(s - 2m^2). \end{aligned} \tag{18}$$

Consequently, the right hand side of eq. (17) becomes

$$\begin{aligned} A_{11} &= -2(t + m^2)^2 - 2(s - 2m^2)t \\ &\quad + 2m^2(s - 2m^2) - \cancel{16m^2 t} + \cancel{8m^2(t + m^2)} + \cancel{8m^2(t + m^2)} - \cancel{16m^4} \\ &= -2(t + m^2)^2 - 2(t - m^2) \times (s - 2m^2 = -t - u) \\ &= 2tu - 6tm^2 + 2um^2 - 2m^4 \\ &= 2(t - m^2)\,(u - 3m^2) - 8m^4. \end{aligned} \tag{19}$$

This completes our evaluation of the first trace.

Now consider the second trace A_{22}. Instead of working through the calculation, we may use the photon exchange / crossing symmetry between the two diagrams (1). This symmetry exchanges $t \leftrightarrow u$ and also $A_{11} \leftrightarrow A_{22}$, thus

$$A_{22} = 2(u - m^2)\,(t - 3m^2) - 8m^4. \tag{20}$$

Traceology 2

Now we need to evaluate the third trace A_{12} which accounts for the interference between the two diagrams (1). This trace is more complicated, so let's start by simplifying the $\gamma^\nu \cdots \gamma_\nu$ part. Back in homework #6, we had

$$\gamma^\nu \not{a} \gamma_\nu = -2\not{a}, \qquad \gamma^\nu \not{a}\not{b}\gamma_\nu = 4(ab), \qquad \gamma^\nu \not{a}\not{b}\not{c}\gamma_\nu = -2\not{c}\not{b}\not{a}, \tag{21}$$

which now gives us

$$\gamma^\nu(\not{q} + m)\gamma^\mu(\not{p}_- + m)\gamma_\nu = -2m^2\gamma^\mu + 4m(q + p_-)^\mu - 2\not{p}_-\gamma^\mu\not{q}. \tag{22}$$

Plugging this formula into eq. (14) for the A_{12} and remembering that we need an even number of slash-momentum factors inside the trace, we obtain

$$\begin{aligned}
A_{12} &= \tfrac{1}{4}\,\mathrm{Tr}\Big(\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_- + m)\gamma_\nu \times (\not{\tilde{q}}+m)\gamma_\mu(\not{p}_+ - m)\Big) \\
&= m(q+p_-)^\mu \times \mathrm{Tr}\Big(m\gamma_\mu\,\not{p}_+ - \not{\tilde{q}}\gamma_\mu m\Big) \\
&\quad -\ \tfrac{1}{2}\,\mathrm{Tr}\Big((m^2\gamma^\mu + \not{p}_-\gamma^\mu\,\not{q}) \times (\not{\tilde{q}}\gamma_\mu\,\not{p}_+ \ -\ m^2\gamma_\mu)\Big) \\
&= m(q+p_-)^\mu \times 4m(p_+ - \tilde{q})_\mu \\
&\quad -\ \tfrac{1}{2}\,\mathrm{Tr}\Big(\not{p}_-\gamma^\mu\,\not{q}\,\not{\tilde{q}}\gamma_\mu\,\not{p}_+ \ -\ m^2\,\not{p}_-\gamma^\mu\,\not{q}\gamma_\mu \ +\ m^2\gamma^\mu\,\not{\tilde{q}}\gamma_\mu\,\not{p}_+ \ -\ m^4\gamma^\mu\gamma_\mu\Big) \\
&= 4m^2(q+p_-)^\mu(p_+ - \tilde{q})_\mu \\
&\quad -\ \tfrac{1}{2}\,\mathrm{Tr}\Big(4(q\tilde{q})\,\not{p}_-\,\not{p}_+ \ +\ 2m^2\,\not{p}_-\,\not{q} \ -\ 2m^2\,\not{\tilde{q}}\,\not{p}_+ \ -\ 4m^4\Big) \\
&= 4m^2\Big(-(q\tilde{q}) \ +\ (qp_+) \ -\ (\tilde{q}p_-) \ +\ (p_-p_+)\Big) \\
&\quad -\ 8(q\tilde{q})(p_-p_+) \ -\ 4m^2(p_-q) \ +\ 4m^2(\tilde{q}p_+) \ +\ 8m^4.
\end{aligned} \tag{23}$$

Finally, we need to work out the kinematics. Besides eqs. (18), we have

$$\begin{aligned}
\tilde{q}p_- &= (p_- - k_2)p_- \ =\ m^2 \ -\ k_2p_- \ =\ m^2 \ +\ \tfrac{1}{2}(u-m^2) \ =\ +\tfrac{1}{2}(u+m^2), \\
\tilde{q}p_+ &= (k_1 - p_+)p_+ \ =\ k_1p_+ \ -\ m^2 \ =\ -\tfrac{1}{2}(u-m^2) \ -\ m^2 \ =\ -\tfrac{1}{2}(u+m^2), \\
\tilde{q}q &= (p_- - k_2)(p_- - k_1) \ =\ p_-^2 \ -\ p_-(k_1+k_2 = p_- + p_+) \ +\ k_1k_2 \\
&= k_1k_2 \ -\ p_-p_+ \ =\ \tfrac{1}{2}s \ -\tfrac{1}{2}(s-2m^2) \ =\ m^2.
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
A_{12} &= 4m^2 \times \Big(-m^2 \ -\ \tfrac{1}{2}(t+m^2) \ -\ \tfrac{1}{2}(u+m^2) \ +\ \tfrac{1}{2}(s-2m^2)\Big) \\
&\quad -\ 4m^2(s-2m^2) \ -\ 2m^2(t+m^2) \ -\ 2m^2(u+m^2) \ +\ 8m^4 \\
&= -2m^2 \times (2t + 2u + s) \\
&= -2m^2 \times (t + u + 2m^2) \\
&= -2m^2(t-m^2) \ -\ 2m^2(u-m^2) \ -\ 8m^4.
\end{aligned} \tag{25}$$

Annihilation Summary

Having worked out the traces, let's plug them into eq. (13):

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} \;&=\; \frac{e^4}{(t-m^2)^2}\times\Big(2(t-m^2)(u-3m^2)\;-\;8m^4\Big)\\
&\quad+\;\frac{e^4}{(u-m^2)^2}\times\Big(2(u-m^2)(t-3m^2)\;-\;8m^4\Big)\\
&\quad+\;\frac{2e^4}{(t-m^2)(u-m^2)}\times\Big(-2m^2(t-m^2)\;-\;2m^2(u-m^2)\;-\;8m^4\Big)\\
&=\;2e^4\left(\frac{u-3m^2}{t-m^2}+\frac{t-3m^2}{u-m^2}-\frac{2m^2}{u-m^2}-\frac{2m^2}{t-m^2}\right)\\
&\quad-\;8e^4m^4\left(\frac{1}{t-m^2}+\frac{1}{u-m^2}\right)^2\\
&=\;2e^4\Bigg[\frac{u-m^2}{t-m^2}+\frac{t-m^2}{u-m^2}-4m^2\left(\frac{1}{t-m^2}+\frac{1}{u-m^2}\right)\\
&\qquad\qquad-\;4m^4\left(\frac{1}{t-m^2}+\frac{1}{u-m^2}\right)^2\Bigg],
\end{aligned}
\tag{26}
$$

or more compactly

$$
\overline{|\mathcal{M}|^2}\;=\;2e^4\left[\frac{u-m^2}{t-m^2}+\frac{t-m^2}{u-m^2}+1\;-\;\left(1+\frac{2m^2}{t-m^2}+\frac{2m^2}{u-m^2}\right)^2\right]. \tag{27}
$$

This is our final result; the rest is kinematics.

In the center of mass frame, $p^\mu_\mp=(E,\pm\mathbf{p})$ where $E=+\sqrt{\mathbf{p}^2+m^2}$, and $k^\mu_{1,2}=(\omega,\pm\mathbf{k})$ where $\omega=|\mathbf{k}|=E$. Consequently,

$$
\begin{aligned}
s\;&=\;4E^2,\\
t\;&=\;-(\mathbf{p}-\mathbf{k})^2\;=\;-\mathbf{p}^2-E^2+2|\mathbf{p}|E\cos\theta,\\
u\;&=\;-(\mathbf{p}+\mathbf{k})^2\;=\;-\mathbf{p}^2-E^2-2|\mathbf{p}|E\cos\theta,\\
t-m^2\;&=\;-2E(E-|\mathbf{p}|\cos\theta),\\
u-m^2\;&=\;-2E(E+|\mathbf{p}|\cos\theta),
\end{aligned}
\tag{28}
$$

and therefore

$$
\begin{aligned}
\frac{u-m^2}{t-m^2}+\frac{t-m^2}{u-m^2}+1 &= \frac{E+|\mathbf{p}|\cos\theta}{E-|\mathbf{p}|\cos\theta}+\frac{E-|\mathbf{p}|\cos\theta}{E+|\mathbf{p}|\cos\theta}+1\\
&= \frac{3E^2+\mathbf{p}^2\cos^2\theta}{E^2-\mathbf{p}^2\cos^2\theta}\\
&= \frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta},\\
\frac{1}{t-m^2}+\frac{1}{u-m^2} &= \frac{-1}{2E}\left(\frac{1}{E-|\mathbf{p}|\cos\theta}+\frac{1}{E+|\mathbf{p}|\cos\theta}\right)\\
&= \frac{-1}{2E}\times\frac{2E}{E^2-\mathbf{p}^2\cos^2\theta} = \frac{-1}{m^2+\mathbf{p}^2\sin^2\theta},\\
1+\frac{2m^2}{t-m^2}+\frac{2m^2}{u-m^2} &= \frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}.
\end{aligned}
\tag{29}
$$

Thus

$$
\overline{|\mathcal{M}|^2} = 2e^4\left[\frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta} - \left(\frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\right)^2\right], \tag{30}
$$

and finally the partial cross section of annihilation

$$
\frac{d\sigma(e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} = \frac{|\mathbf{k}|}{|\mathbf{p}|}\frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} = \frac{\alpha^2}{8E|\mathbf{p}|}\times\left[\frac{3m^2+\mathbf{p}^2(3+\cos^2\theta)}{m^2+\mathbf{p}^2\sin^2\theta} - \left(\frac{\mathbf{p}^2\sin^2\theta-m^2}{\mathbf{p}^2\sin^2\theta+m^2}\right)^2\right]. \tag{31}
$$

For the non-relativistic electron and positron with $|\mathbf{p}| \ll m$, the expression in the square brackets becomes $3-(-1)^2=2$, hence *isotropic* partial cross section

$$
\frac{d\sigma(\text{slow } e^+e^-\to\gamma\gamma)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4m|\mathbf{p}|}. \tag{32}
$$

And the total cross section in this limit is

$$
\sigma_{\text{tot}}(\text{slow } e^+e^-\to\gamma\gamma) = \frac{4\pi}{2}\times\frac{\alpha^2}{4m|\mathbf{p}|} = \frac{\pi\alpha^2}{2m|\mathbf{p}|}, \tag{33}
$$

where total solid angle is $4\pi/2$ because of 2 identical photons in the final state.

In the opposite limit of ultra-relativistic e^- and e^+ with $|\mathbf{p}| \approx E \gg m$, we have

$$\Big[\cdots\Big] \;\approx\; \frac{3+\cos^2\theta}{\sin^2\theta} \;-\; 1 \;=\; \frac{2(1+\cos^2\theta)}{\sin^2\theta} \tag{34}$$

and hence highly un-isotropic cross section

$$\frac{d\sigma(\text{fast } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} \;\approx\; \frac{\alpha^2}{4E^2} \times \frac{1+\cos^2\theta}{\sin^2\theta}\,. \tag{35}$$

Note how this cross-section is strongly peaked in the forward direction $\theta = 0$ where one photon continues the electron's motion while the other continues the positron's motion.

According to eq. (35), the total annihilation cross-section

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) \;=\; 2\pi \int\limits_0^{\pi/2} d\theta\, \sin\theta\, \frac{d\sigma}{d\Omega_{\text{cm}}} \tag{36}$$

diverges at small angles, but that's an artefact of the approximation (34) becoming inaccurate at small angles where $\mathbf{p}^2 \sin^2\theta \lesssim m^2$. Instead, for small angles we have

$$\Big[\cdots\Big] \;=\; \frac{4\mathbf{p}^2}{m^2 + \mathbf{p}^2\theta^2} \;+\; O(1) \tag{37}$$

and consequently

$$\frac{d\sigma(\text{fast } e^+e^- \to \gamma\gamma)}{d\Omega_{\text{c.m.}}} \;\approx\; \frac{\alpha^2}{4E^2} \times \frac{2\mathbf{p}^2}{m^2 + \mathbf{p}^2\theta^2}\,. \tag{38}$$

This cross-section is strongly peaked in the forward direction, but it does not diverge. Instead,

$$\sigma_{\text{tot}}(\text{fast } e^+e^- \to \gamma\gamma) \;=\; \frac{\pi\alpha^2}{E^2} \times \left(\log\frac{2E}{m} \;-\; \frac{1}{2}\right). \tag{39}$$