1. Consider a massive relativistic vector field $A^{\mu}(x)$ with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_{\mu} A^{\mu} - A^{\mu} J_{\mu}$$
(1)

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and the current $J^{\mu}(x)$ is a fixed source for the $A^{\mu}(x)$ field. Note that because of the mass term, the Lagrangian (1) is not gauge invariant.

- (a) Derive the Euler-Lagrange field equations for the massive vector field $A^{\mu}(x)$.
- (b) Show that this field equation does not require current conservation; however, if the current happens to satisfy $\partial_{\mu}J^{\mu} = 0$, then the field $A^{\mu}(x)$ satisfies

$$\partial_{\mu}A^{\mu} = 0$$
 and $(\partial^2 + m^2)A^{\mu} = J^{\mu}.$ (2)

2. According to the Noether theorem, a translationally invariant system of classical fields ϕ_a has a conserved stress-energy tensor

$$T_{\text{Noether}}^{\mu\nu} = \sum_{a} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi_{a})} \partial^{\nu}\phi^{a} - g^{\mu\nu}\mathcal{L}.$$
 (3)

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}} + \partial_{\lambda} \mathcal{K}^{[\lambda\mu]\nu}, \qquad (4)$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3-index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$,

$$\partial_{\mu}T^{\mu\nu} = \partial_{\mu}T^{\mu\nu}_{\text{Noether}} = (\text{hopefully}) = 0$$

$$P^{\mu}_{\text{net}} \equiv \int d^{3}\mathbf{x} T^{0\mu} = \int d^{3}\mathbf{x} T^{0\mu}_{\text{Noether}}.$$
(5)

For the scalar fields, real or complex, $T_{\text{Noether}}^{\mu\nu}$ is properly symmetric and one simply has $T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu}$. Unfortunately, the situation is more complicated for the vector, tensor or

spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_{\mu}, \partial_{\nu}A_{\mu}) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$
(6)

where A_{μ} is a real vector field and $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

- (b) Write down $T_{\text{Noether}}^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.
- (c) The properly symmetric and also gauge invariant stress-energy tensor for the free electromagnetism is

$$T_{\rm EM}^{\mu\nu} = -F^{\mu\lambda}F^{\nu}_{\lambda} + \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}.$$
 (7)

Show that this expression indeed has form (4) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

(d) Write down the components of the stress-energy tensor (7) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Now consider the electromagnetic fields coupled to the electric current J^{μ} of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^{\mu}_{\rm EM}$ and $P^{\mu}_{\rm mat}$. Consequently, we should have

$$\partial_{\mu}T_{\text{net}}^{\mu\nu} = 0 \quad \text{for} \quad T_{\text{net}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} \tag{8}$$

but generally $\partial_{\mu}T_{\rm EM}^{\mu\nu} \neq 0$ and $\partial_{\mu}T_{\rm mat}^{\mu\nu} \neq 0$.

(e) Use Maxwell's equations to show that

$$\partial_{\mu}T_{\rm EM}^{\mu\nu} = -F^{\nu\lambda}J_{\lambda} \tag{9}$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_{λ} according to

$$\partial_{\mu}T_{\rm mat}^{\mu\nu} = +F^{\nu\lambda}J_{\lambda}.$$
 (10)