1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{1}$$

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\rm def}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad\text{and}\qquad (\partial^2 + m^2)A^\mu = J^\mu. \tag{2}$$

2. According to the Noether theorem, a translationally invariant system of classical fields ϕ_a has a conserved stress-energy tensor

$$T^{\mu\nu}_{\rm Noether} \;=\sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{3}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\rm Noether} \;+\; \partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}, \tag{4}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$,

$$\begin{aligned} \partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\rm Noether} \;=\text{(hopefully)}=\; 0\\ P^\mu_{\rm net} \;&\equiv\; \int\! d^3\mathbf{x}\,T^{0\mu} \;=\; \int\! d^3\mathbf{x}\,T^{0\mu}_{\rm Noether}\,. \end{aligned} \tag{5}$$

For the scalar fields, real or complex, $T^{\mu\nu}_{\rm Noether}$ is properly symmetric and one simply has $T^{\mu\nu} = T^{\mu\nu}_{\rm Noether}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \tag{6}$$

where A_μ is a real vector field and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) Write down $T^{\mu\nu}_{\text{Noether}}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\text{EM}} \;=\; -F^{\mu\lambda} F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\, g^{\mu\nu}\, F_{\kappa\lambda} F^{\kappa\lambda}. \tag{7}$$

Show that this expression indeed has form (4) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

(d) Write down the components of the stress-energy tensor (7) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Now consider the electromagnetic fields coupled to the electric current J^μ of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P^μ_{EM} and P^μ_{mat}. Consequently, we should have

$$\partial_\mu T^{\mu\nu}_{\text{net}} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\text{net}} \;=\; T^{\mu\nu}_{\text{EM}} \;+\; T^{\mu\nu}_{\text{mat}} \tag{8}$$

but generally $\partial_\mu T^{\mu\nu}_{\text{EM}} \neq 0$ and $\partial_\mu T^{\mu\nu}_{\text{mat}} \neq 0$.

(e) Use Maxwell's equations to show that

$$\partial_\mu T^{\mu\nu}_{\text{EM}} \;=\; -F^{\nu\lambda} J_\lambda \tag{9}$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda} J_\lambda. \tag{10}$$