1. Continuing the previous homework set, consider a classical theory made of a complex scalar field Φ of charge $q \neq 0$ and the the EM fields:

$$\mathcal{L}_{\text{net}} = D^\mu\Phi^* D_\mu\Phi - m^2\Phi^*\Phi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1}$$

where

$$D_\mu\Phi = (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu\Phi^* = (\partial_\mu - iqA_\mu)\Phi^* \tag{2}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \stackrel{\text{def}}{=} -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{3}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole field system and show that

$$T^{\mu\nu}_{\text{net}} \equiv T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{mat}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{4}$$

where

$$T^{\mu\nu}_{\text{EM}} = -F^{\mu\alpha}F^{\nu}_{\ \alpha} + \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{5}$$

as for the free EM,

$$K^{\lambda\mu\nu} \equiv -K^{\mu\lambda\nu} = -F^{\lambda\mu}A^\nu, \tag{6}$$

also exactly as for the free EM, and

$$T^{\mu\nu}_{\text{mat}} = D^\mu\Phi^*\, D^\nu\Phi + D^\nu\Phi^*\, D^\mu\Phi - g^{\mu\nu}\big(D_\lambda\Phi^*\, D^\lambda\Phi - m^2\Phi^*\,\Phi\big). \tag{7}$$

Note: In the presence of an electric current J^μ, the $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (7) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{8}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_\lambda \tag{9}$$

and therefore the *net* stress-energy tensor (4) is conserved.

Note: the last statement follows from problem 1.2 (e). Do not redo that problem here, just quote the result.

2. Next, consider the quantum electromagnetic fields. Canonical quantization of the massless vector field $A_\mu(x)$ is rather difficult because of the redundancy associated with the gauge symmetry, so let me simply state without proof a few key properties of the quantum tension fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$. In the absence of electric charges and currents, these fields satisfy time-independent operatorial identities

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t) \;= \nabla\cdot\hat{\mathbf{B}}(\mathbf{x},t) \;=\; 0 \tag{10}$$

and have equal-time commutation relations

$$\begin{aligned}
\big[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{x}',t'=t)\big] \;&=\; 0,\\
\big[\hat{B}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\big] \;&=\; 0,\\
\big[\hat{E}_i(\mathbf{x},t), \hat{B}_j(\mathbf{x}',t'=t)\big] \;&=\; -i\hbar c\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta^{(3)}(\mathbf{x}-\mathbf{x}').
\end{aligned} \tag{11}$$

(a) Verify that the commutation relations (11) are consistent with the time-independent Maxwell equations (10).

In the Heisenberg picture, the quantum EM fields also obey the time-dependent Maxwell equations

$$\begin{aligned} \frac{\partial \hat{\mathbf{B}}}{\partial \mathbf{t}} &= -\nabla \times \hat{\mathbf{E}}\,, \\ \frac{\partial \hat{\mathbf{E}}}{\partial \mathbf{t}} &= +\nabla \times \hat{\mathbf{B}}\,. \end{aligned} \tag{12}$$

(b) Derive eqs. (12) from the free electromagnetic Hamiltonian

$$\hat{H}_{EM} \;=\; \int\! d^3\mathbf{x}\, \left(\tfrac{1}{2}\hat{\mathbf{E}}^2 + \tfrac{1}{2}\hat{\mathbf{B}}^2\right) \tag{13}$$

and the equal-time commutation relations (11).

3. Finally, let us quantize a complex scalar field $\Phi(x)$. For simplicity, let's restrict to a free relativistic field, thus classically

$$\mathcal{L} \;=\; \partial^\mu \Phi^* \,\partial_\mu \Phi \;-\; m^2\, \Phi^* \Phi. \tag{14}$$

In the Hamiltonian formalism, we trade the time derivatives $\partial_0 \Phi(x)$ and $\partial_0 \Phi^*(x)$ for the canonically conjugate fields $\Pi(x)$ and $\Pi^*(x)$. (Note that for complex field $\Pi(\mathbf{x})$ is canonically conjugate to the $\Phi^*(\mathbf{x})$ while $\Pi^*(\mathbf{x})$ is canonically conjugate to the $\Phi(\mathbf{x})$.) Canonical quantization of this system yields non-hermitian quantum fields $\hat{\Phi}(x) \neq \hat{\Phi}^\dagger(x)$ and $\hat{\Pi}(x) \neq \hat{\Pi}^\dagger(x)$ and the Hamiltonian operator

$$\hat{H} \;=\; \int\! d^3\mathbf{x}\, \left(\hat{\Pi}^\dagger \hat{\Pi} \,+\, \nabla \hat{\Phi}^\dagger \cdot \nabla \hat{\Phi} \,+\, m^2\, \hat{\Phi}^\dagger \hat{\Phi}\right). \tag{15}$$

(a) Derive the Hamiltonian (15) and write down the equal-time commutation relations between the quantum fields $\hat{\Phi}(\mathbf{x})$, $\hat{\Phi}^\dagger(\mathbf{x})$, $\hat{\Pi}(\mathbf{x})$ and $\hat{\Pi}^\dagger(\mathbf{x})$.

Next, let us expand the quantum fields into plane-wave modes:

$$\hat{\Phi}(\mathbf{x}) \;=\; \sum_{\mathbf{p}} L^{-3/2} e^{i\mathbf{x}\mathbf{p}} \hat{\Phi}_{\mathbf{p}}, \qquad \hat{\Phi}_{\mathbf{p}} \;=\; \int\! d^3\mathbf{x}\, L^{-3/2} e^{-i\mathbf{p}\mathbf{x}}\, \hat{\Phi}(\mathbf{x}), \tag{16}$$

and ditto for the $\hat{\Phi}^\dagger(\mathbf{x})$, $\hat{\Pi}(\mathbf{x})$, and $\hat{\Pi}^\dagger(\mathbf{x})$ fields. Note that for the *non–hermitian* fields $\hat{\Phi}^\dagger_{\mathbf{p}} \neq \hat{\Phi}_{-\mathbf{p}}$ and $\hat{\Pi}^\dagger_{\mathbf{p}} \neq \hat{\Pi}_{-\mathbf{p}}$; instead, all the mode operators $\hat{\Phi}_{\mathbf{p}}$, $\hat{\Phi}^\dagger_{\mathbf{p}}$, $\hat{\Pi}_{\mathbf{p}}$, and $\hat{\Pi}^\dagger_{\mathbf{p}}$ are completely independent of each other. Consequently, we have two independent species of creation and annihilation operators, *i.e.* for each mode $\mathbf{p}$ we have independent operators

$$\begin{aligned} &\hat{a}_{\mathbf{p}} \stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{\mathbf{p}} + i\hat{\Pi}_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \qquad \hat{a}^\dagger_{\mathbf{p}} \stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}^\dagger_{\mathbf{p}} - i\hat{\Pi}^\dagger_{\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \\ \text{and} \\ &\hat{b}_{\mathbf{p}} \stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}^\dagger_{-\mathbf{p}} + i\hat{\Pi}^\dagger_{-\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \qquad \hat{b}^\dagger_{\mathbf{p}} \stackrel{\text{def}}{=} \frac{E_{\mathbf{p}}\hat{\Phi}_{-\mathbf{p}} - i\hat{\Pi}_{-\mathbf{p}}}{\sqrt{2E_{\mathbf{p}}}}, \end{aligned} \tag{17}$$

where $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$.

(b) Verify the bosonic commutation relations (at equal times) between the annihilation operators $\hat{a}_{\mathbf{p}}$ and $\hat{b}_{\mathbf{p}}$ and the corresponding creation operators $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{b}^\dagger_{\mathbf{p}}$.

(c) Show that the Hamiltonian of the free charged fields is

$$\hat{H} = \int d^3\mathbf{x} \left(\Pi^\dagger\,\Pi + \nabla\Phi^\dagger\cdot\nabla\Phi + m^2\Phi^\dagger\,\Phi\right) = \sum_{\mathbf{p}} \left(E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + E_{\mathbf{p}}\hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right) + \text{const.} \tag{18}$$

Next, consider the classical Noether current

$$J^\mu = i\Phi^*\partial^\mu\Phi - i(\partial^\mu\Phi^*)\Phi. \tag{19}$$

of the global $U(1)$ symmetry $\Phi(x) \mapsto e^{i\theta}\Phi(x)$. In the Hamiltonian formalism

$$J^0 = i\Phi^*\Pi - i\Pi^*\Phi,$$

so its quantization suffers from the operator ordering ambiguity. To resolve the ambiguity, we *define* the charge density operator in the quantum theory as

$$\hat{\rho}(\mathbf{x}) = \tfrac{i}{2}\left\{\hat{\Pi}^\dagger(\mathbf{x}), \hat{\Phi}(\mathbf{x})\right\} - \tfrac{i}{2}\left\{\hat{\Pi}(\mathbf{x}), \hat{\Phi}^\dagger(\mathbf{x})\right\}. \tag{20}$$

(d) Show that in terms of creation and annihilation operators, the net charge operator

$\hat{Q} = \int d^3\mathbf{x}\,\hat{\rho}(\mathbf{x})$ becomes

$$\hat{Q} \;=\; \sum_{\mathbf{p}} \left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;-\; \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right). \tag{21}$$

Finally, consider the stress-energy tensor of the charged field. Classically, Noether theorem gives

$$T^{\mu\nu} \;=\; \partial^\mu\Phi^*\,\partial^\nu\Phi \;+\; \partial^\mu\Phi\,\partial^\nu\Phi^* \;-\; g^{\mu\nu}\mathcal{L}. \tag{22}$$

Quantization of this formula is straightforward (modulo ordering ambiguity); for example, $\hat{\mathcal{H}} \equiv \hat{T}^{00}$ is precisely the integrand on the right hand side of eq. (15).

(e) Show that the total mechanical momentum operator of the fields is

$$\hat{\mathbf{P}}_{\text{mech}} \;\stackrel{\text{def}}{=}\; \int d^3\mathbf{x}\,\hat{T}^{0,\mathbf{i}} \;=\; \sum_{\mathbf{p}} \left(\mathbf{p}\,\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;+\; \mathbf{p}\,\hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}\right) \tag{23}$$

Physically, eqs. (23), (18) and (21) show that a complex field $\Phi(x)$ describes a relativistic particle together with its antiparticle; they have exactly the same rest mass m but exactly opposite charges ± 1.