1. An operator acting on identical bosons can be described in terms of N–particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

   First consider the one-body operators, *i.e.* additive operators acting on one particle at a time. In the first-quantized formalism they act on N–particle states according to

$$\hat{A}^{(1)}_{\rm tot} \;=\; \sum_{i=1}^{N} \hat{A}_1(i^{\underline{\rm th}}\ \text{particle}) \tag{1}$$

   where $\hat{A}_1$ is some kind of a one-particle operator (such as momentum $\hat{\mathbf{p}}$, or kinetic energy $\frac{1}{2m}\hat{\mathbf{p}}^2$, or potential $V(\hat{\mathbf{x}})$, *etc.*, *etc.*). In the second-quantized formalism such operators become

$$\hat{A}^{(2)}_{\rm tot} \;=\; \sum_{\alpha,\beta} \langle\alpha|\,\hat{A}_1\,|\beta\rangle\ \hat{a}^\dagger_\alpha \hat{a}_\beta\,. \tag{2}$$

   (a) Your task is to show that for any one-particle operator $\hat{A}_1$ and any N–particle state $|\Psi\rangle$,

$$\hat{A}^{(1)}_{\rm tot}\,|\Psi\rangle \;=\; \hat{A}^{(1)}_{\rm tot}\,|\Psi\rangle\,. \tag{3}$$

   For simplicity, you should first prove this equality for $\hat{A}_1 = |\alpha\rangle\langle\beta|$ and $|\Psi\rangle = |\gamma_1,\ldots,\gamma_N\rangle$. Then you can use linearity to generalize to any N–particle states $|\Psi\rangle$ and any one-particles operators $\hat{A}_1$. Note: $\hat{A}_1 = \sum_{\alpha,\beta}|\alpha\rangle\langle\alpha|\,\hat{A}_1\,|\beta\rangle\langle\beta|$.

   Next, consider two-body operators, *i.e.* additive operators acting on two particle at a time. Given a two-particle operator $\hat{B}_2$ — such as $V(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)$ — the *total* B operator acts in the first-quantized formalism according to

$$\hat{B}^{(1)}_{\rm tot} \;=\; \tfrac{1}{2}\sum_{i\neq j} \hat{B}_2(i^{\underline{\rm th}}\ \text{and}\ j^{\underline{\rm th}}\ \text{particles}), \tag{4}$$

and in the second-quantized formalism according to

$$\hat{B}^{(2)}_{\text{tot}} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} \left(\left(\langle\alpha|\otimes\langle\beta|\right)\hat{B}_2\left(|\gamma\rangle\otimes|\delta\rangle\right)\right) \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta \,. \tag{5}$$

(b) Again, show that for any 2–particle operator $\hat{B}_2$ and any $N \geq 2$ particle state $|\Psi\rangle$,

$$\hat{B}^{(1)}_{\text{tot}}\,|\Psi\rangle \;=\; \hat{B}^{(1)}_{\text{tot}}\,|\Psi\rangle \,. \tag{6}$$

2. Next, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; 0, \qquad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \qquad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta} \,. \tag{7}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$.

(b) Consider three one-particle operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$. Let us define the corresponding second-quantized operators $\hat{A}^{(2)}_{\text{tot}}$, $\hat{B}^{(2)}_{\text{tot}}$, and $\hat{C}^{(2)}_{\text{tot}}$ according to eq. (2).

Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C}^{(2)}_{\text{tot}} = \left[\hat{A}^{(2)}_{\text{tot}}, \hat{B}^{(2)}_{\text{tot}}\right]$.

(c) Next, calculate the commutator $[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]$.

(d) Finally, let $\hat{A}_1$ be a one-particle operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}^{(2)}_{\text{tot}}$, $\hat{B}^{(2)}_{\text{tot}}$, and $\hat{C}^{(2)}_{\text{tot}}$ be the corresponding second-quantized operators according to eqs. (2) and (5).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\underline{\text{st}}}) + \hat{A}_1(2^{\underline{\text{nd}}})\right), \hat{B}_2\right]$ then $\hat{C}^{(2)}_{\text{tot}} = \left[\hat{A}^{(2)}_{\text{tot}}, \hat{B}^{(2)}_{\text{tot}}\right]$.

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$.

(a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\text{def}}{=} \exp\!\left(\xi\hat{a}^\dagger - \xi^*\hat{a}\right)|0\rangle$. Show that

$$|\xi\rangle \;=\; e^{-|\xi|^2/2}\, e^{\xi\hat{a}^\dagger}\,|0\rangle \quad \text{and} \quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle \,. \tag{8}$$

(b) Calculate the uncertainties Δq and Δp for a coherent state $|\xi\rangle$ and verify their minimality: $\Delta q \Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\text{def}}{=} \langle\hat{n}\rangle = |\xi|^2$.

Hint: use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$.

(c) Consider time-dependent coherent states $|\xi(t)\rangle$. Show that for $\xi(t) = \xi_0 e^{-i\omega t}$, the state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}\,|\xi(t)\rangle = \hat{H}\,|\xi(t)\rangle$.

(d) The coherent states are not quite orthogonal to each other. Calculate their overlap $\langle\eta|\xi\rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(e) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{9}$$

for any given classical complex field $\Phi(\mathbf{x})$.

(f) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \;\stackrel{\text{def}}{=}\; \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{10}$$

(g) Let

$$\hat{H} \;=\; \int d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger\cdot\nabla\hat{\Psi} \;+\; V(\mathbf{x})\hat{\Psi}^\dagger\hat{\Psi}\right)$$

and show that for any classical field configuration $\Phi(\mathbf{x},t)$ that satisfies the classical field equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{11}$$

(h) Finally, show that the quantum overlap $|\,\langle\Phi_1|\Phi_2\rangle\,|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.