1. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and $\pi^0$, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2) \cong \mathrm{Spin}(4)$ which would be exact if the two lightest quark flavors $u$ and $d$ were exactly massless; in reality, the current quark masses $m_u$ and $m_d$ do not exactly vanish but are small enough to be treated as a perturbation. Exact or approximate, the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken Spin(4)/Spin(3) give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and $\pi^0$.

   QCD is a rather complicated theory, so it is often convenient to describe the physics of the spontaneously broken chiral symmetry in terms of a simpler effective theory such as the linear sigma model. This model has 4 real scalar fields; in terms of the unbroken isospin symmetry, we have an isosinglet $\sigma(x)$ and an isotriplet $\underset{\sim}{\pi}(x)$ comprising $\pi^1(x)$, $\pi^2(x)$ and $\pi^3(x)$ (or equivalently, $\pi^0(x) \equiv \pi^3(x)$ and $\pi^\pm(x) \equiv \big(\pi^1(x) \pm i\pi^2(x)\big)/\sqrt{2}$). The Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu\sigma)^2 \;+\; \tfrac{1}{2}(\partial_\mu\underset{\sim}{\pi})^2 \;-\; \frac{\lambda}{8}\left(\sigma^2 + \underset{\sim}{\pi}^2 - f^2\right)^2 \;+\; \beta\sigma \tag{1}$$

   is invariant under the $SO(4)$ rotations of the four fields, except for the last term which we take to be very small. (In QCD $\beta \sim \frac{m_u+m_d}{2f}\left\langle\overline{\Psi}\Psi\right\rangle$ which is indeed very small because the $u$ and $d$ quarks are very light.)

   In class, we discussed this theory for $\beta = 0$ and showed that it has $SO(4)$ spontaneously broken to $SO(3)$ and hence 3 massless Goldstone bosons. In this exercise, we let $\beta > 0$ but $\beta \ll \lambda f^3$ to show how this leads to massive but light pions.

(a) Show that the scalar potential of the linear sigma model with $\beta > 0$ has a unique minimum at

$$\langle \underset{\sim}{\pi} \rangle \;=\; 0 \quad \text{and} \quad \langle \sigma \rangle \;=\; f \;+\; \frac{\beta}{\lambda f^2} \;+\; O(\beta^2). \tag{2}$$

(b) Expand the fields around this minimum and show that the pions are light while the $\sigma$ particle is much heavier. Specifically, $M_\pi^2 \approx (\beta/f)$ while $M_\sigma^2 \approx \lambda f^2$.

2. The rest of this homework is about the Bogolyubov transform and the superfluid helium. Let us start with some kind of annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ which satisfy the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] \;=\; [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; 0, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \;=\; \delta_{\mathbf{k},\mathbf{k}'}\,. \tag{3}$$

Let us define new operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ according to

$$\hat{b}_{\mathbf{k}} \;=\; \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\hat{a}^\dagger_{-\mathbf{k}}\,, \quad \hat{b}^\dagger_{\mathbf{k}} \;=\; \cosh(t_{\mathbf{k}})\hat{a}^\dagger_{\mathbf{k}} \;+\; \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}} \tag{4}$$

for some arbitrary real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$.

(a) Show that the $\hat{b}_{\mathbf{k}}$ and the $\hat{b}^\dagger_{\mathbf{k}}$ satisfy the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and the $\hat{a}^\dagger_{\mathbf{k}}$.

The Bogolyubov transform — replacing the 'original' creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ with the 'transformed' operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ — is useful for diagonalizing quadratic Hamiltonians of the form

$$\hat{H} \;=\; \sum_{\mathbf{k}} A_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;+\; \tfrac{1}{2} \sum_{\mathbf{k}} B_{\mathbf{k}} \left( \hat{a}_{\mathbf{k}} \hat{a}_{-\mathbf{k}} \;+\; \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} \right) \tag{5}$$

where for all momenta $\mathbf{k}$, $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$.

(b) Show that for a suitable choice of the $t_{\mathbf{k}}$ parameters,

$$\hat{H} \;=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;+\; \text{const} \qquad \text{where } \omega_{\mathbf{k}} \;=\; \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}\,. \tag{6}$$

(c) Show that $\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} - \hat{b}^\dagger_{-\mathbf{k}} \hat{b}_{-\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}} \hat{a}_{-\mathbf{k}}$ and therefore

$$\hat{\mathbf{P}} \;\equiv\; \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} \;=\; \sum_{\mathbf{k}} \mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\,. \tag{7}$$

3. Now consider the quantum field theory of superfluid helium. As discussed in class, we start with a semi-classical ground state, which is the coherent state with a symmetry-breaking expectation value $\left\langle \hat{\Psi}(\mathbf{x}) \right\rangle \equiv \sqrt{n} = \sqrt{\mu/\lambda}$. In class, we shifted the classical field $\Phi(x)$ by $\sqrt{n}$, but here we shift the *quantum* field $\hat{\Psi}(x)$ by the same amount, thus

$$\hat{\Psi}(\mathbf{x}) \;=\; \sqrt{n} \,+\, \delta\hat{\Psi}(\mathbf{x}), \qquad \hat{\Psi}^\dagger(\mathbf{x}) \;=\; \sqrt{n} \,+\, \delta\hat{\Psi}^\dagger(\mathbf{x}). \tag{8}$$

(a) Rewrite the Hamiltonian (or rather the free-energy operator) in terms of the shifted quantum fields and show that

$$\hat{H} - \mu\hat{N} \;=\; \text{const} \;+\; \hat{H}_{\text{free}} \;+\; \hat{H}_{\text{int}}\,, \tag{9}$$

where

$$\hat{H}_{\text{free}} \;=\; \int d^3\mathbf{x} \left\{ \frac{1}{2M} \nabla\delta\hat{\Psi}^\dagger \cdot \nabla\delta\hat{\Psi} \;+\; \frac{\lambda n}{2} \left( \delta\hat{\Psi}^\dagger\delta\hat{\Psi}^\dagger + 2\delta\hat{\Psi}^\dagger\delta\hat{\Psi} + \delta\hat{\Psi}\delta\hat{\Psi} \right) \right\} \tag{10}$$

is quadratic with respect to the shifted fields while $\hat{H}_{\text{int}}$ comprises the cubic and the quartic terms.

Now let us Fourier-expand the shifted quantum fields in terms of the shifted annihilation and creation operators $\tilde{a}_{\mathbf{k}} = \hat{a}_{\mathbf{k}} - \sqrt{N}\delta_{\mathbf{k},\mathbf{0}}$ and $\tilde{a}^\dagger_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} - \sqrt{N}\delta_{\mathbf{k},\mathbf{0}}$.

(b) Expand the free Hamiltonian (10) in terms of $\tilde{a}_{\mathbf{k}}$ and $\tilde{a}^{\dagger}_{\mathbf{k}}$, then apply the Bogolyubov transform to obtain

$$\hat{H}_{\text{free}} \;=\; \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}^{\dagger}_{\mathbf{k}} \hat{b}_{\mathbf{k}} \;+\; \text{const} \tag{11}$$

where

$$\omega_{\mathbf{k}} \;=\; |\mathbf{k}| \times \sqrt{\frac{\lambda n}{M} \,+\, \frac{\mathbf{k}^2}{4M^2}}. \tag{12}$$

Now, let us allow for a more general model of liquid helium in which the two-body forces between the atoms are not delta-like, $V_2(\mathbf{x}-\mathbf{y}) \neq \lambda\delta^{(3)}(\mathbf{x}-\mathbf{y})$.

(c) Write the free energy operator for this model, then proceed as in parts (a) and (b) to obtain the free quasiparticle Hamiltonian (11). This time, you should get

$$\omega_{\mathbf{k}} \;=\; |\mathbf{k}| \times \sqrt{\frac{n}{M}\,\tilde{V}(\mathbf{k}) \,+\, \frac{\mathbf{k}^2}{4M^2}} \qquad \text{where} \quad \tilde{V}(\mathbf{k}) \;=\; \int\! d^3\mathbf{x}\, e^{i\mathbf{k}\mathbf{x}} V_2(\mathbf{x}). \tag{13}$$

Now consider the *quasiparticles* created by the $\hat{b}^{\dagger}_{\mathbf{k}}$ operators and annihilated by the $\hat{b}_{\mathbf{k}}$. The quasiparticle vacuum is the unique state $|\Omega\rangle$ annihilated by all the $\hat{b}_{\mathbf{k}}$ operators, $\hat{b}_{\mathbf{k}}\,|\Omega\rangle \;=\; 0$. This state is the ground state of the Hamiltonian (11), while the excited states have several quasiparticles, $|QP : \mathbf{k}_1, \ldots, \mathbf{k}_n\rangle \propto \hat{b}^{\dagger}_{\mathbf{k}_n} \cdots \hat{b}^{\dagger}_{\mathbf{k}_1}\,|\Omega\rangle$. Note that according to eq. (7), the quasiparticles have definite mechanical momenta. On the other hand, they do not have well-defined atomic numbers. This is related to the spontaneous breakdown of the phase symmetry, which is generated by the atom number operator $\hat{N}$. Physically, the quasi-particles interpolate between phonons in the superfluid (for small $\mathbf{k}$) and atoms knocked out of the Bose condensate (for large $\mathbf{k}$) — note the appropriate limits of the dispersion relations (12) and (13).

(d) Check that for large momenta $\hat{b}^{\dagger}_{\mathbf{k}} \approx \hat{a}^{\dagger}_{\mathbf{k}}$ and therefore the quasi-particle is approximately an atom, while for small momenta $\hat{b}^{\dagger}_{\mathbf{k}} \approx (\text{coeff}) \times (\hat{a}^{\dagger}_{\mathbf{k}} + \hat{a}_{\mathbf{k}})$ and therefore the quasiparticle is approximately a phonon.

Finally, let us consider a moving superfluid. For simplicity, we assume uniform motion with velocity $\mathbf{v}$. As discussed in class, this motion is described by

$$\phi(\mathbf{x}) \equiv \left\langle \hat{\Psi}(\mathbf{x}) \right\rangle = \sqrt{n} \times \exp(iM\mathbf{v} \cdot \mathbf{x}), \tag{14}$$

so let's define the shifted quantum fields according to

$$\hat{\Psi}(\mathbf{x}) = e^{iM\mathbf{vx}} \times \left(\sqrt{n} + \delta\hat{\Psi}(\mathbf{x})\right), \qquad \hat{\Psi}^\dagger(\mathbf{x}) = e^{-iM\mathbf{vx}} \times \left(\sqrt{n} + \delta\hat{\Psi}^\dagger(\mathbf{x})\right). \tag{15}$$

Physically, the $e^{\pm iM\mathbf{vx}}$ factors multiplying the shifted fields mean that the latter describe fluctuations in the frame of the moving superfluid rather than in the lab frame.

(e) Write the free-energy operator of the moving superfluid in terms of the shifted fields and show that

$$\hat{H} - \mu'\hat{N} = \text{const} + \hat{H}_{\text{free}} + \mathbf{v} \cdot \hat{\mathbf{P}} + \hat{H}_{\text{int}} \tag{16}$$

where $\hat{H}_{\text{free}}$ and $\hat{H}_{\text{int}}$ are exactly as for the superfluid at rest, and the momentum operator is exactly as in eq. (7). Note that the chemical potential here includes the kinetic energy of the uniform motion, $\mu' = \mu + \frac{1}{2}M\mathbf{v}^2$.

From the moving fluid's point of view, the free Hamiltonian of the quasiparticles comprises

$$\hat{H}'_{\text{free}} = \hat{H}_{\text{free}} + \mathbf{v} \cdot \hat{\mathbf{P}} = \sum_{\mathbf{k}} (\omega_{\mathbf{k}} + \mathbf{v} \cdot \mathbf{k})\, \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}\,. \tag{17}$$

Given the dispersion relation (13), as long as the flow speed $|\mathbf{v}|$ is less then some critical speed

$$v_c = \min_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{|\mathbf{k}|} > 0, \tag{18}$$

coefficients of all the $\hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}$ terms in the Hamiltonian (17) are positive. Consequently, there is no spontaneous creation of quasiparticles, and hence no dissipation of the superflow. *This is why the superfluid flows without resistance.*