1. The first problem explains why quantum mechanics of a fixed number of relativistic particles does not work (except as an approximation). Indeed, consider a free relativistic spinless particle with Hamiltonian

$$\hat{H} \;=\; +\sqrt{M^2+\hat{\mathbf{P}}^2} \tag{1}$$

(in the $c=\hbar=1$ units). In the coordinate picture, this Hamiltonian is a horrible integro–differential operator, but that's only a technical problem. The real problem concerns the time evolution kernel

$$U(\mathbf{x}-\mathbf{y};t) \;=\; \langle\mathbf{x},t|\mathbf{y},t_0=0\rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle\mathbf{x}|\exp(-it\hat{H})\,|\mathbf{y}\rangle^{\text{Schroedinger}}_{\text{picture}}\;. \tag{2}$$

(a) Show that

$$U(\mathbf{x}-\mathbf{y};t) \;=\; \frac{-i}{4\pi^2\,r}\int\!dk\,k\,\exp\bigl(irk-it\omega(k)\bigr), \tag{3}$$

where $r=|\mathbf{x}-\mathbf{y}|$ and $\omega(k)=\sqrt{M^2+k^2}$.

(b) Let us take the limit $t\to\infty$, $r\to\infty$, with fixed ratio r/t; let's stay inside the future light cone, so $(r/t)<1$. Show that in this limit, the evolution kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \;\approx\; \frac{(-iM)^{3/2}}{4\pi^{3/2}}\,\frac{t}{(t^2-r^2)^{5/4}}\times\exp(-iM\sqrt{t^2-r^2}). \tag{4}$$

Hint: Use the saddle point method to evaluate the integral (3). If you are not familiar with this method, see the mathematical supplement.

(c) Now let's take a similar approximation but go outside the light cone, thus fixed $(r/t)>1$ while $R,t\to\infty$. Show that in this limit, the kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \;\approx\; \frac{iM^{3/2}}{4\pi^{3/2}}\,\frac{t}{(r^2-t^2)^{5/4}}\times\exp(-M\sqrt{r^2-t^2}). \tag{5}$$

This formula shows that the kernel diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{y}$ at the time $t_0=0$, after time $t > 0$, its wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' lightcones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x. I will show in class that the two effects *precisely* cancel each other, so altogether there is no propagation outside the light cone. That's how relativistic QFT is perfectly causal while the relativistic QM is not.

2. Back in homework#2, problem 2, I introduced the free quantum EM fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$, their commutation relations, and the Hamiltonian. In this problem, you shall connect those fields to photon creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$.

   As a first step, let us decompose Schrödinger-picture fields into Fourier modes,

$$\hat{\mathbf{E}}(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=\pm1} e^{i\mathbf{kx}}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{E}_{\mathbf{k},\lambda}\,, \qquad \hat{\mathbf{B}}(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=\pm1} e^{i\mathbf{kx}}\mathbf{e}_{\mathbf{k},\lambda}\,\hat{B}_{\mathbf{k},\lambda}\,, \tag{6}$$

   where for each $\mathbf{k}$, the $\mathbf{e}_{\mathbf{k},\pm1}$ are two unit vectors perpendicular to $\mathbf{k}$ and to each other. Note that we need $\mathbf{e}_{\mathbf{k},\lambda}\perp\mathbf{k}$ to assure transversality of the free EM fields, $\nabla\cdot\hat{\mathbf{E}} = \nabla\cdot\hat{\mathbf{B}} = 0$.

   For future convenience, let us use the helicity basis of polarizations in which the unit vectors $\mathbf{e}_{\mathbf{k},\pm}$ are eigenvectors of the cross product with $\mathbf{k}$,

$$\mathbf{k}\times\mathbf{e}_{\mathbf{k},\lambda} = i\lambda|\mathbf{k}|\,\mathbf{e}_{\mathbf{k},\lambda}\,, \quad \lambda = \pm1, 0. \tag{7}$$

   The $\lambda = 0$ polarization is longitudinal (parallel to the $\mathbf{k}$) while the $\lambda = \pm1$ polarizations are transverse ($\perp\mathbf{k}$). Note that the transverse polarization vectors $\mathbf{e}_{\mathbf{k},\pm1}$ are complex, so orthogonality and unit lengths mean

$$\mathbf{e}^*_{\mathbf{k},\lambda}\cdot\mathbf{e}_{\mathbf{k},\lambda'} \;=\; \delta_{\lambda,\lambda'}\,. \tag{8}$$

To fix all the important phases, we let $\mathbf{e}_{\mathbf{k},0}=\mathbf{k}/|\mathbf{k}|$, while for the transverse polarizations we require

$$\mathbf{e}^*_{\mathbf{k},\lambda} \;=\; \mathbf{e}_{\mathbf{k},-\lambda} \;=\; \mathbf{e}_{-\mathbf{k},+\lambda} \quad \text{for } \lambda=\pm1 \qquad \text{and} \quad \mathbf{e}_{\mathbf{k},+1}\times\mathbf{e}_{\mathbf{k},-1} \;=\; +i\frac{\mathbf{k}}{|\mathbf{k}|}\,. \tag{9}$$

(a) Work out the equal-time commutation relations for the $\hat{E}_{\mathbf{k},\lambda}$ and $\hat{B}_{\mathbf{k},\lambda}$ operators. Also, show that

$$\hat{H} \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_{\lambda=\pm1}\left(\tfrac{1}{2}\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;+\; \tfrac{1}{2}\hat{B}^\dagger_{\mathbf{k},\lambda}\hat{B}_{\mathbf{k},\lambda}\right). \tag{10}$$

(b) Define the photonic creation and annihilation operators according to

$$\begin{aligned}\hat{a}_{\mathbf{k},\lambda} \;&=\; \lambda\hat{B}_{\mathbf{k},\lambda} \;+\; i\hat{E}_{\mathbf{k},\lambda}\,,\\ \hat{a}^\dagger_{\mathbf{k},\lambda} \;&=\; \lambda\hat{B}^\dagger_{\mathbf{k},\lambda} \;-\; i\hat{E}^\dagger_{\mathbf{k},\lambda}\,,\end{aligned} \tag{11}$$

and show that they satisfy the relativistically-normalized bosonic commutation relations. Also show that

$$\hat{H} \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_{\lambda=\pm1}\omega_\mathbf{k}\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \text{zero point energy} \tag{12}$$

where $\omega_\mathbf{k}=|\mathbf{k}|$.

(c) Express the Heisenberg-picture fields $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$ in terms of the Schrödinger-picture creation and annihilator operators. In relativistic notations you should get

$$\hat{F}^{\mu\nu}(x) \;=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\sum_{\lambda=\pm1}\left(e^{-ikx}\,f^{\mu\nu}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} \;+\; e^{+ikx}\,(f^{\mu\nu}_{\mathbf{k},\lambda})^*\,\hat{a}^\dagger_{\mathbf{k},\lambda}\right)_{k^0=+\omega_\mathbf{k}} \tag{13}$$

for some polarization tensors $f^{\mu\nu}_{\mathbf{k},\lambda}$.

(d) Now consider the quantum vector field $\hat{A}^{\mu}(x)$. Show that in the Heisenberg picture

$$\hat{A}^{\mu}(x) \;=\; -\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\sum_{\lambda=\pm1}\Big(e^{-ikx}\,e^{\mu}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda}\;+\;e^{+ikx}\,(e^{\mu}_{\mathbf{k},\lambda})^*\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}\Big)_{k^0=+\omega_{\mathbf{k}}} \tag{14}$$

where $e^{\mu}_{\mathbf{k},\lambda} = (0,\mathbf{e}_{\mathbf{k},\lambda}) + k^{\mu}C_{\mathbf{k},\lambda}$ for some gauge-dependent some coefficients $C_{\mathbf{k},\lambda}$; in the Coulomb gauge ($\nabla\cdot\hat{\mathbf{A}}(x)\equiv 0$) $C_{\mathbf{k},\lambda}=0$ and $e^{\mu}_{\mathbf{k},\lambda}=(0,\mathbf{e}_{\mathbf{k},\lambda})$.

Assume that the gauge condition on the $\hat{A}^{\mu}(x)$ is linear and local, and does not allow for degrees of freedom not contained in the quantum $\hat{F}^{\mu\nu}$ fields (*i.e.*, for operators not made out of the $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$).

3. Finally, consider the EM propagators.

(a) First, a lemma: Show that

$$\sum_{\lambda=\pm1} f^{\mu\nu}_{\mathbf{k},\lambda}\left(f^{\alpha\beta}_{\mathbf{k},\lambda}\right)^* \;=\; -k^{\mu}k^{\alpha}g^{\nu\beta}\;-\;k^{\nu}k^{\beta}g^{\mu\alpha}\;+\;k^{\mu}k^{\beta}g^{\nu\alpha}+\;k^{\nu}k^{\alpha}g^{\mu\beta} \tag{15}$$

where $k^0=\omega_{\mathbf{k}}=|\mathbf{k}|$.

(b) Next, another lemma: Show that in any gauge consistent with eq. (14),

$$\sum_{\lambda=\pm1} e^{\mu}_{\mathbf{k},\lambda}\left(e^{\alpha}_{\mathbf{k},\lambda}\right)^* \;=\; -g^{\mu\alpha}\;+\;k^{\mu}q^{*\alpha}(k)\;+\;q^{\mu}(k)k^{\alpha} \tag{16}$$

for some gauge-dependent 4-vector $q^{\mu}(k)$.

(c) Next, show that

$$\langle 0|\,\hat{A}^{\mu}(x)\,\hat{A}^{\alpha}(y)\,|0\rangle \;=\;\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega}\Big[\Big(-g^{\mu\alpha}\;+\;k^{\mu}q^{*\alpha}(k)\;+\;q^{\mu}(k)k^{\alpha}\Big)e^{-ik(x-y)}\Big]_{k_0=+\omega_{\mathbf{k}}} \tag{17}$$

(d) Finally, the Feynman propagator: Show that

$$G^{\mu\alpha}_F(x-y)\;\equiv\;\langle 0|\,\mathbf{T}^*\hat{A}^{\mu}(x)\hat{A}^{\alpha}(y)\,|0\rangle\;=\;\int\frac{d^4\mathbf{k}}{(2\pi)^4}\,\left(-g^{\mu\alpha}+k^{\mu}q^{*\alpha}+q^{\mu}k^{\alpha}\right)\,\frac{ie^{-ik(x-y)}}{k^2+i0} \tag{18}$$

for some gauge-dependent $q^\mu(k)$. In the Feynman gauge, $q^\mu \equiv 0$ and hence

$$G_F^{\mu\alpha}(x-y) \;=\; -g^{\mu\alpha} \times D_F(x-y). \tag{19}$$

For the vector fields, the time-ordered product $\mathbf{T}$ is modified to

$$\mathbf{T}^* \hat{A}^\mu(x)\hat{A}^\nu(y) \;=\; \mathbf{T}\hat{A}^\mu(x)\hat{A}^\nu(y) \;+\; i\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y). \tag{20}$$

For the explanation of this modification, please see *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.