1. First, an exercise in Dirac matrices γ^μ. Please do not assume any specific form of these 4×4 matrices, just use the anti-commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1}$$

In class, we have defined the spin matrices

$$S^{\mu\nu} = -S^{\nu\mu} \stackrel{\rm def}{=} \tfrac{i}{4}\left[\gamma^\mu,\gamma^\nu\right] \tag{2}$$

and showed that

$$\left[S^{\mu\nu},\gamma^\lambda\right] = ig^{\nu\lambda}\gamma^\mu - ig^{\mu\lambda}\gamma^\nu. \tag{3}$$

(a) Show that the spin matrices $S^{\mu\nu}$ have commutation relations of the Lorentz generators,

$$\left[S^{\kappa\lambda},S^{\mu\nu}\right] = ig^{\lambda\mu}S^{\kappa\nu} - ig^{\lambda\nu}S^{\kappa\mu} - ig^{\kappa\mu}S^{\lambda\nu} + ig^{\kappa\nu}S^{\lambda\mu}. \tag{4}$$

Continuous Lorentz transforms obtain from integrating infinite sequences of infinitesimal transforms $X'^\mu = X^\mu + \epsilon\Theta^\mu{}_\nu X^\nu$ where $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$. Altogether, a finite continuous transform acts as $X'^\mu = L^\mu_\nu X^\nu$ where

$$L = \exp(\Theta), \quad i.e., \quad L^\mu_{\ \nu} = \delta^\mu_\nu + \Theta^\mu{}_\nu + \tfrac{1}{2}\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu + \tfrac{1}{6}\Theta^\mu{}_\kappa\Theta^\kappa{}_\lambda\Theta^\lambda{}_\nu + \cdots. \tag{5}$$

(b) Let L be a Lorentz transform of the form (5), and let $M(L) = \exp\left(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}\right)$. Show that $M^{-1}(L)\gamma^\mu M(L) = L^\mu_\nu\gamma^\nu$.

Next, a little more algebra:

(c) Calculate $\{\gamma^\rho,\gamma^\lambda\gamma^\mu\gamma^\nu\}$, $[\gamma^\rho,\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$ and $[S^{\rho\sigma},\gamma^\lambda\gamma^\mu\gamma^\nu]$.

(d) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$ and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$. Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

A charged spinor field $\Psi(x)$ in an EM background $A^\mu(x)$ satisfies gauge-covariant version of the Dirac equation, namely $\big(i\gamma^\mu D_\mu + m\big)\Psi(x) = 0$ where $D_\mu = \partial_\mu + iqA_\mu(x)$ are the covariant derivatives.

(e) Show that the this equation implies $\big(m^2 + D^2 + qF_{\mu\nu}S^{\mu\nu}\big)\Psi(x) = 0$.

2. Next, consider the $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ matrix.

   (a) Show that γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices $S^{\mu\nu}$.

   (b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

   (c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

   (d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

   (e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$ and γ^5.

Conventions: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$, $\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$,
$\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \frac{1}{6}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda)$,
and ditto for the $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}$.

Now consider Dirac matrices in spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (1), but their sizes depend on d.

Let $\Gamma = (\pm1 \text{ or } \pm i)\gamma^0\gamma^1\cdots\gamma^{d-1}$ be the generalization of the γ^5 to d dimensions; the pre-factor ±1 or $\pm i$ is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

   (f) *For even d*, Γ anticommutes with all the γ^μ. Prove this, and use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

   (g) *For odd d*, Γ commutes with all the Γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

   Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

3. *Parity* is an im-proper Lorentz transform. In $3+1$ dimensions, it reflects the space coordinates but not the time,

$$\mathcal{P} : (\mathbf{x}, t) \mapsto (-\mathbf{x}, t). \tag{6}$$

(a) Parity acts on Dirac spinor fields according to

$$\Psi'(\mathbf{x}, t) \;=\; \pm\gamma^0 \, \Psi(-\mathbf{x}, t) \tag{7}$$

where the overall $\pm$ sign is the *intrinsic parity* of a particular Dirac field.

Verify that the Dirac equation is covariant under this transformation and that the Dirac action $\int d^4x \, \mathcal{L}_{\text{Dirac}}$ is invariant.

In other *even* spacetime dimensions, parity acts as in eq. (6) and the Dirac fields transform according to (7). But in the *odd* spacetime dimension — *i.e.*, even space dimensions — the reflection of all space coordinates at once is a combination of 180° rotations. Instead, parity reflects just one space coordinate,

$$\mathcal{P} : (t, x^1, x^2, \ldots, x^{d-1}) \mapsto (+t, -x^1, +x^2, \ldots, x^{d-1}). \tag{8}$$

(b) How does a Dirac spinor field $\Psi(x)$ in an odd spacetime dimension transform under parity? Show that for a massless field, the Dirac action is invariant under parity, but for a massive field, the mass term in the Lagrangian breaks the symmetry.

Finally, consider a massless Dirac field $\Psi(x)$ coupled to a real scalar field $\Phi(x)$,

$$\mathcal{L} \;=\; i\overline{\Psi}\partial\!\!\!/\Psi \;+\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \tfrac{1}{2}M_s^2\Phi^2 \;+\; g\Phi\overline{\Psi}\Psi. \tag{9}$$

(c) Show that the action $\int d^dx \, \mathcal{L}$ is parity-invariant provided Φ is a true scalar — $\Phi'(x') = +\Phi(x)$ — for even d, and pseudoscalar — $\Phi'(x') = -\Phi(x)$ — for odd d.