1. Consider bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where $\Gamma$ is one of 16 matrices discussed in the previous homework; altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}i\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad \text{and} \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{1}$$

   (a) Show that all the bilinears (1) are Hermitian.
   Hint: First, show that $\left(\overline{\Psi}\Gamma\Psi\right)^\dagger = \overline{\Psi}\overline{\Gamma}\Psi$.
   Note: despite the Fermi statistics, $\left(\Psi_\alpha^\dagger\Psi_\beta\right)^\dagger = +\Psi_\beta^\dagger\Psi_\alpha$.

   (b) Show that under *continuous* Lorentz symmetries, the $S$ and the $P$ transform as scalars, the $V^\mu$ and the $A^\mu$ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

   (c) Find the transformation rules of the bilinears (1) under parity (*cf.* problem 2 of the previous set) and show that while $S$ is a true scalar and $V$ is a true (polar) vector, $P$ is a pseudoscalar and $A$ is an axial vector.

   Next, consider the charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take $\Psi(x)$ and $\Psi^\dagger(x)$ to be "classical" fermionic fields which *anticommute* with each other, $\Psi_\alpha\Psi_\beta^\dagger = -\Psi_\beta^\dagger\Psi_\alpha$.

   (d) In the Weyl convention, $\mathcal{C} : \Psi(x) \mapsto \pm\gamma^2\Psi^*(x)$. Show that $\mathcal{C} : \overline{\Psi}\Gamma\Psi \mapsto \overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

   (e) Calculate $\Gamma^c$ for all 16 independent matrices $\Gamma$ and find out which Dirac bilinears are $\mathcal{C}$–even and which are $\mathcal{C}$–odd.

   (f) Verify that the Dirac action is invariant under the charge conjugation.

2. Next, a few exercises concerning the plane-wave solutions $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,x)$ of the Dirac equation.

   (a) Verify that $u^\dagger(p,s)u(p,s') = 2p^0\delta_{s,s'}$ and likewise $v^\dagger(p,s)v(p,s') = 2p^0\delta_{s,s'}$. Also, show that for $p' = (+p^0, -\mathbf{p})$, $u^\dagger(p,s)v(p',s') = 0$.

(b) Show that $\gamma^0 u(p,s) = +u(p',s)$ but $\gamma^0 v(p,s) = -v(p',s)$ where $p' = (p^0, -\mathbf{p})$. Also show that in the Weyl basis, $\gamma^2 u^*(p,s) = v(p,s)$ and $\gamma^2 v^*(p,s) = u(p,s)$.

(c) Show that

$$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta}\,. \tag{2}$$

(d) Prove the Gordon identity

$$\bar{u}(p',s')\gamma^\mu u(p.s) \;=\; \frac{(p'+p)^\mu}{2m}\,\bar{u}(p's')u(p,s) \;+\; \frac{i(p'-p)_\nu}{m}\,\bar{u}(p's')S^{\mu\nu}u(p,s). \tag{3}$$

Note: This time, the momenta $p$ and $p'$ are unrelated to each other.

Hint: First, use Dirac equations for the $u$ and the $\bar{u}'$ to show that $2m\bar{u}'\gamma^\mu u = \bar{u}'(\not{p}'\gamma^\mu + \gamma^\mu\not{p})u$.

(e) Generalize the Gordon identity to $\bar{u}'\gamma^\mu v$, $\bar{v}'\gamma^\mu u$ and $\bar{v}'\gamma^\mu v$.

3. Now consider the quantum Dirac fields

$$\begin{aligned}
\hat{\Psi}(x) \;&=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \left(e^{-ipx}u(p,s)\,\hat{a}_{p,s} \;+\; e^{+ipx}v(p,s)\,\hat{b}^\dagger_{p,s}\right),\\
\hat{\Psi}^\dagger(x) \;&=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{2E_{\mathbf{p}}}\sum_s \left(e^{+ipx}u^\dagger(p,s)\,\hat{a}^\dagger_{p,s} \;+\; e^{-ipx}v^\dagger(p,s)\,\hat{b}^\dagger_{p,s}\right),
\end{aligned} \tag{4}$$

where $\hat{a}^\dagger$, $\hat{b}^\dagger$ and $\hat{a}$, $\hat{b}$ are relativistically-normalized fermionic creation and annihilation operators. Those operators satisfy the anti-commutation relations

$$\begin{aligned}
\left\{\hat{a}^\dagger_{p,s}, \hat{a}_{p',s'}\right\} \;&=\; \left\{\hat{b}^\dagger_{p,s}, \hat{b}_{p',s'}\right\} \;=\; 2E_{\mathbf{p}}\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\delta_{s,s'}\,,\\
\left\{\hat{a}\,\text{or}\,\hat{b}, \hat{a}\,\text{or}\,\hat{b}\right\} \;&=\; \left\{\hat{a}^\dagger\,\text{or}\,\hat{b}^\dagger, \hat{a}^\dagger\,\text{or}\,\hat{b}^\dagger\right\} \;=\; \left\{\hat{a}, \hat{b}^\dagger\right\} \;=\; \left\{\hat{a}^\dagger, \hat{b}\right\} \;=\; 0,
\end{aligned} \tag{5}$$

but you don't need them for this exercise.

In the Fock space, the charge conjugation operator $\hat{\mathcal{C}} = \hat{\mathcal{C}}^{-1}$ acts as $\hat{\mathcal{C}}\,|q,\mathbf{p},s\rangle = \pm\,|-q,\mathbf{p},s\rangle$ where $q$ is the charge (which have opposite signs for particles and antiparticles) and the overall $\pm$ sign is the same for all particles and antiparticles of a given species; it's called the *intrinsic* C. Consequently, $\hat{\mathcal{C}}$ transforms the creation and annihilation operators according to

$$\begin{aligned} \hat{\mathcal{C}}\hat{a}_{p,s}\hat{\mathcal{C}} &= \pm\hat{b}_{p,s}\,, \qquad & \hat{\mathcal{C}}\hat{a}^\dagger_{p,s}\hat{\mathcal{C}} &= \pm\hat{b}^\dagger_{p,s}\,, \\ \hat{\mathcal{C}}\hat{b}_{p,s}\hat{\mathcal{C}} &= \pm\hat{a}_{p,s}\,, \qquad & \hat{\mathcal{C}}\hat{b}^\dagger_{p,s}\hat{\mathcal{C}} &= \pm\hat{a}^\dagger_{p,s}\,. \end{aligned} \tag{6}$$

(a) Show that eqs. (6) imply that the quantum Dirac field $\hat{\Psi}(x)$ transforms under charge conjugation exactly as in problem **1**, namely $\hat{\mathcal{C}}\hat{\Psi}(x)\hat{\mathcal{C}} = \pm\gamma^2\hat{\Psi}^*(x)$. Here $\hat{\Psi}^*(x)$ means the transpose (in the Dirac matrix sense) of the $\hat{\Psi}^\dagger(x)$, *i.e.* the *column* made of the four $\Psi^\dagger_\alpha(x)$.

Hint: Use what you should have proved in problem **2**(b).

Now consider the parity (space reflection) operator. In the Fock space, it acts as $\hat{\mathcal{P}}\,|q,\mathbf{p},s\rangle = \pm\,|q,-\mathbf{p},s\rangle$ where the overall sign is the *intrinsic parity* of the species in question. Note that parity reverses the direction of the 3–momentum $\mathbf{p}$, but it does not reverse the spin $s$ because $\vec{S}$ is an axial vector. Similar to the charge conjugation, $\hat{\mathcal{P}}^2 = 1$ so $\hat{\mathcal{P}}^{-1} = \hat{\mathcal{P}}$.

(b) The quantum Dirac field should transform under parity as $\hat{\mathcal{P}}\hat{\Psi}(\mathbf{x},t)\hat{\mathcal{P}} = \pm\gamma^0\hat{\Psi}(-\mathbf{x},+t)$, *cf.* the previous homework set. Show that this requires

$$\begin{aligned} \hat{\mathcal{P}}\hat{a}_{\mathbf{p},s}\hat{\mathcal{P}} &= \pm\hat{a}_{-\mathbf{p},s}\,, \qquad & \hat{\mathcal{P}}\hat{a}^\dagger_{\mathbf{p},s}\hat{\mathcal{P}} &= \pm\hat{a}^\dagger_{-\mathbf{p},s}\,, \\ \hat{\mathcal{P}}\hat{b}_{\mathbf{p},s}\hat{\mathcal{P}} &= \mp\hat{b}_{-\mathbf{p},s}\,, \qquad & \hat{\mathcal{P}}\hat{b}^\dagger_{\mathbf{p},s}\hat{\mathcal{P}} &= \mp\hat{b}^\dagger_{-\mathbf{p},s}\,, \end{aligned} \tag{7}$$

where *the particles and the antiparticles have opposite intrinsic parities.*

Hint: Use what you should have proved in problem **2**(b).