1. In a Yukawa theory with $M_s > 2m_f$, the scalar particles are unstable against decay into fermion + antifermion pairs.

   Write down the tree-level matrix element $\mathcal{M}(S \to f + \bar{f})$, sum $|\mathcal{M}|^2$ over final particles' spins, and calculate the total decay rate $S \to f + \bar{f}$.

2. Consider a Yukawa theory of two Dirac fields $\Psi_1(x)$ and $\Psi_2(x)$ coupled to the same real scalar field $\Phi(x)$:

$$\begin{aligned}\mathcal{L} \;&= \overline{\Psi}_1(i\partial\!\!\!/ - m_1)\Psi_1 \;+\; \overline{\Psi}_2(i\partial\!\!\!/ - m_2)\Psi_2 \;+\; \tfrac{1}{2}(\partial_\mu\Phi)^2 \;-\; \tfrac{1}{2}M_s^2\Phi^2 \\ &\quad-\; g_1\Phi\overline{\Psi}_1\Psi_1 \;-\; g_2\Phi\overline{\Psi}_2\Psi_2\,.\end{aligned} \tag{1}$$

   At the tree level, calculate the matrix element, the partial cross-section and the total cross-section for elastic scattering of one type of a fermion off the other type, $f_1 + f_2 \to f_1 + f_2$. Take the initial fermions to be unpolarized and sum over the final fermion's polarizations.

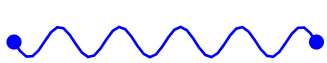
3. Consider a QED-like theory of a *massive* vector field A^μ coupled to a Dirac field Φ,

$$\mathcal{L} \;=\; -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \;+\; \frac{m_v^2}{2}A^\mu A_\mu \;+\; \overline{\Psi}(i\partial\!\!\!/ - M_f)\Psi \;+\; eA^\mu\overline{\Psi}\gamma_\mu\Psi. \tag{2}$$

   The Feynman rules of this theory are similar to those of QED, except for the "photon" propagator: For a massive vector,

$$G_F^{\mu\nu}(x-y) \;\equiv\; \langle 0|\,\mathbf{T}^*\hat{A}^\mu(x)\hat{A}^\nu(y)\,|0\rangle \;=\; -\big(g^{\mu\nu} + \frac{1}{m_v^2}\partial^\mu\partial^\nu\big)G_F^{\text{scalar}}(x-y), \tag{3}$$

   or in momentum space form,

A^μ  A^ν

$$= \frac{i}{q^2 - m_v^2 + i\epsilon}\left(-g^{\mu\nu} \;+\; \frac{q^\mu q^\nu}{m_v^2}\right). \tag{4}$$

   Also, the external lines factors $e^\mu_{\mathbf{k},\lambda}$ and $e^{*\mu}_{\mathbf{k},\lambda}$ for massive vectors have three allowed polarizations rather than two. (But you don't need them for this exercise.)

(a) Draw tree diagrams and write down the tree-level amplitude for the elastic scattering of two fermions in this theory. Simplify the amplitude using

$$\bar{u}(p', s')\not{q}u(p, s) \;=\; 0 \quad \text{for } q = p' - p. \tag{5}$$

(I shall prove this identity in class this week.)

(b) Take the non-relativistic limit $|\mathbf{p}| \ll M_f$ for the external fermions and assume the vector field is relatively light, $m_v \ll M_f$. Simplify the scattering amplitude in this limit, compare it to the Born approximation to the non-relativistic scattering of a potential $V(\mathbf{x} - \mathbf{y})$, and show that it agrees with a Yukawa-like potential

$$V(r) \;=\; +\frac{e^2}{4\pi r}\, e^{-m_v r}. \tag{6}$$

Pay attention to the sign of this potential: vector-mediated forces between like charges are repulsive, not attractive.

Now consider the elastic scattering of a fermion and an antifermion.

(c) Draw the diagrams and write down the exact tree-level amplitude.

(d) Finally, take the non-relativistic limit as in part (b), simplify the amplitude, and show that it agrees with the Born approximation to scattering off the potential

$$V(r) \;=\; -\frac{e^2}{4\pi r}\, e^{-m_v r}. \tag{7}$$

Again, pay attention to the sign of this potential: The fermion and the antifermion have opposite charges, so they attract rather then repel.