1. A muon usually decays into an electron, an electron-flavored antineutrino, and a muon-flavored neutrino, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. In the Fermi theory of weak interactions, the matrix element for this decays is

$$\left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left| \mu^- \right\rangle = \frac{G_F}{\sqrt{2}} \left[\bar{u}(\nu_\mu)(1-\gamma^5)\gamma^\alpha u(\mu^-)\right] \times \left[\bar{u}(e^-)(1-\gamma^5)\gamma_\alpha v(\bar{\nu}_e)\right]. \quad (1)$$

The modern Standard Model of particle interactions produces essentially the same amplitude at the tree level.

Experimentally, the neutrinos and antineutrinos are hard to detect. But it is easy to measure the muon's net decay rate $\Gamma = 1/\tau_\mu$ and the energy distribution $d\Gamma/dE_e$ of the electrons produced by decaying muons. Your task is to calculate these quantities from the amplitude (1).

(a) Sum the absolute square of the amplitude (1) over the final particle spins and average over the initial muon's spin. Show that altogether,

$$\tfrac{1}{2} \sum_{\substack{\text{all}\\\text{spins}}} \left| \left\langle e^-, \bar{\nu}_e, \nu_\mu \right| \mathcal{M} \left| \mu^- \right\rangle \right|^2 = 64 G_F^2 (p_\mu \cdot p_{\bar{\nu}})\,(p_e \cdot p_\nu). \quad (2)$$

The rest of this problem is the phase space calculation. The following lemma is very useful for three-body decays like $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$:

(b) Consider a generic three-body decay of some particles of mass M_0 into three particles of respective masses m_1, m_2, and m_3. Show that in the rest frame of the original particle, the partial decay rate is given by

$$d\Gamma = \frac{1}{2M_0} \times \overline{|\mathcal{M}|^2} \times \frac{d^3\Omega}{256\pi^5} \times dE_1\, dE_2\, dE_3\, \delta(E_1 + E_2 + E_3 - M_0), \quad (3)$$

where $d^3\Omega$ comprises three angular variables parameterizing the directions of the three final-state particles relative to some external frame, but not affecting the angles between the three momenta. For example, one may use two angles to describe the orientation of the decay plane (the three momenta are coplanar, $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$) and one more angle to fix the direction of *e.g.*, $\mathbf{p}_1$ in that plane. Altogether, $\int d^3\Omega = 4\pi \times 2\pi = 8\pi^2$.

Also show that when $m_1 = m_2 = m_3 = 0$, the kinematically allowed range of the final particles' energies is given by

$$0 \leq E_1, E_2, E_3 \leq \tfrac{1}{2}M_0 \qquad \text{while} \quad E_1 + E_2 + E_3 = M_0, \tag{4}$$

but for the non-zero masses $m_{1,2,3}$ this range is much more complicated.

The electron and the neutrinos are much lighter then the muon, so in most decay events all three final-state particles are ultra-relativistic. This allows us to approximate $m_e \approx m_\nu \approx m_{\bar{\nu}} \approx 0$, which greatly simplifies the last part of this exercise:

(c) Integrate the muon's partial decay rate over the final particle energies and derive first $d\Gamma/dE_e$ and then the total decay rate.

2. Now consider the *Bhabha scattering* $e^-e^+ \to e^-e^+$. In QED, there are two tree-level Feynman diagrams contributing to this process. Note that their contributions must be added *before* squaring the amplitude and adding/averaging over spins,

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\Re\big(\mathcal{M}_1^*\mathcal{M}_2\big) \neq |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2. \tag{5}$$

Your task is to calculate the un-polarized partial cross-section $d\sigma/d\Omega$ for the Bhabha scattering. For simplicity, assume $E \gg m_e$ and neglect the electron's mass throughout your calculation. You may find it convenient to use Mandelstam's Lorentz-invariant kinematic variables s, t, and u, see eq. (5.69) of the Peskin & Schroeder textbook for details. Note $s + t + u = 4m_e^2 \approx 0$.

The answer to this problem is simple:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{\alpha^2}{2s}\left[\left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2 + \left(\frac{u}{s} + \frac{u}{t}\right)^2\right] \tag{6}$$

but the intermediate steps are quite complicated, so beware.