1. Consider a QFT where heavy (*i.e.*, $M_s \gg m_e$) neutral scalar particles have Yukawa–like coupling to electrons, which in turn couple to photons according to the usual QED rules, thus

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\not{D} - m_e)\Psi + \left[\tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}M_s^2\varphi^2\right] + g\varphi\times\overline{\Psi}\Psi. \qquad (1)$$

In this theory, an electron and a positron colliding with energy $E_{\text{c.m.}} > M_s$ may annihilate into one photon and one scalar particle, $e^- + e^+ \to \gamma + S$.

(a) Draw tree diagrams for the $e^- + e^+ \to \gamma + S$ process and write down the tree-level matrix element $\langle\gamma S|\,\mathcal{M}\,|e^-e^+\rangle$.

(b) Verify the Ward identity for the photon.

(c) Sum $|\mathcal{M}|^2$ over the photon's polarizations, average over the fermion's spins, and calculate the partial cross-section

$$\frac{d\sigma(e^-e^+ \to \gamma S)}{d\Omega_{\text{c.m.}}}.$$

For simplicity, neglect the electron's mass. But don't neglect the mass of the scalar.

Note: because of the scalar's mass, the kinematic relations between Mandelstam's s, t, and u and between momentum products such as $(k_\gamma p_\mp)$, *etc.*, are different from the $e^+e^- \to \gamma\gamma$ annihilation.

2. Now consider the Compton scattering of a photon off an electron — $\gamma e^- \to \gamma e^-$ — at the tree level of pure QED.

(a) Use crossing symmetry between the Compton scattering and the $e^+e^- \to \gamma\gamma$ annihilation — and my notes for the latter — to show that for the Compton scattering

$$\frac{1}{4}\sum_{\lambda,\lambda'}\sum_{s,s'}|\mathcal{M}|^2 = 2e^4\left[-\frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2} - 1 + \left(1 + \frac{2m^2}{s-m^2} + \frac{2m^2}{u-m^2}\right)^2\right] \qquad (2)$$

The Compton scattering is usually observed in the *lab frame* where the electron is initially at rest. So let's work out the lab frame kinematics.

(b) Use energy and momentum conservation to show that in the lab frame

$$\frac{1}{\omega'} = \frac{1}{\omega} + \frac{1-\cos\theta}{m_e}. \tag{3}$$

Note that for any fixed scattering angle $\theta \neq 0$, the energy of the scattered photon can never exceed $m_e/(1-\cos\theta)$, regardless of the initial photon's energy.

(c) Evaluate the phase-space factor in the lab frame and show that

$$\frac{d\sigma}{d\Omega_{\text{lab}}} = \frac{\omega'^2}{64\pi^2 m_e^2 \omega^2} \times \overline{|\mathcal{M}|^2}. \tag{4}$$

(d) Evaluate $s - m^2$ and $u - m^2$ in the lab frame and derive the Klein–Nishina formula:

$$\frac{d\sigma(\text{Compton})}{d\Omega_{\text{lab}}} = \frac{\alpha^2}{2m_e^2} \times \frac{\omega'^2}{\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right]. \tag{5}$$

(e) Show that for low-energy photons $\omega \ll m_e$, the Klein–Nishina formula agrees with the classical Thompson scattering of an EM wave off a free charged non-relativistic particle.

(f) Finally, show that for very high energy photons with $\omega \gg m_e$,

$$\frac{d\sigma}{d\Omega_{\text{lab}}} = \frac{\alpha^2}{2m_e\omega} \times \frac{1}{1-\cos\theta}, \tag{6}$$

except for very small angles $\theta \lesssim \sqrt{m/\omega}$ where

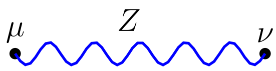
$$\frac{d\sigma}{d\Omega_{\text{lab}}} = \frac{\alpha^2}{2m_e\omega} \times \frac{2}{\theta^2 + (2m/\omega)} \times \begin{cases} 1 & \text{for } \theta \gg \sqrt{2m/\omega}\,, \\ O(1) & \text{for } \theta \lesssim \sqrt{2m/\omega}\,, \end{cases} \tag{7}$$

and the total Compton cross-section is

$$\sigma_{\text{tot}} = \frac{\pi\alpha^2}{m_e\omega} \times \left[\log\frac{\omega}{m_e} + O(1)\right]. \tag{8}$$

3. Finally, consider the $e^+e^- \to \mu^+\mu^-$ pair production in the Standard Model of electroweak interactions. In pure QED, there is only one tree-level Feynman diagram contributing to this process, but in the Standard Model there are two more: one with a virtual Z^0 vector in the s channel, and one with a virtual Higgs scalar. Fortunately, the Higgs field has very small couplings to electrons and muons, so its contribution may be neglected. But the Z^0's contribution becomes important at high energies $E_{\rm cm} \gtrsim O(M_z)$.

The Z^0 particle is a quantum of a massive neutral vector field Z^μ; its propagator is



$$= \frac{i}{q^2 - M_Z^2 + i\epsilon} \times \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{M_Z^2} \right). \tag{9}$$

The Z^μ field couples to the charged leptons (e, μ, τ) according to

$$\mathcal{L} \supset g' Z_\mu \times \sum_{i=e,\mu,\tau} \overline{\Psi}_i \gamma^\mu \left(\sin^2\theta_w - \frac{1-\gamma^5}{4} \right) \Psi_i \,, \tag{10}$$

where

$$g' = \frac{e}{\sin\theta_w \, \cos\theta_w} \tag{11}$$

and θ_w is the Weinberg's weak mixing angle; experimentally $\sin^2\theta_w = 0.232$.

(a) Write down the combined tree-level amplitude $\mathcal{M}(e^+e^- \to \mu^+\mu^-)$ due to both virtual photon and virtual Z.

(b) Assume both the electrons and the muons to be ultra-relativistic ($E_{\rm c.m.} = O(M_Z) \gg m_\mu, m_e$) and evaluate the amplitude (a) for all possible particle helicities. (Use the center-of-mass frame.)

Hint:

$$\gamma^\mu \left(g_V + g_A \gamma^5 \right) = g_L \, \gamma^\mu \, \frac{1-\gamma^5}{2} + g_R \, \gamma^\mu \, \frac{1+\gamma^5}{2} \tag{12}$$

where $g_L = g_V - g_A$ and $g_R = g_V + g_A$.

(c) Calculate the total cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ and the forward-backward asymmetry

$$A = \frac{\sigma(\theta < \pi/2) - \sigma(\theta > \pi/2)}{\sigma(\theta < \pi/2) + \sigma(\theta > \pi/2)} \tag{13}$$

as functions of the total energy $E_{\text{c.m.}}$. For simplicity, approximate $\sin^2\theta_w \approx \frac{1}{4}$ and hence $g_V \approx 0$.

Note that in QED the tree-level pair production is symmetric with respect to $\theta \to \pi - \theta$; the asymmetry in the Standard Model arises from the interference between the virtual-photon and virtual-Z diagrams.