Please do not waste time and paper by copying the posted homework solutions or supplementary notes. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. Likewise, don't re-derive anything I derived in class.

1. The first problem is about chiral symmetry breaking. Consider a theory of four left-handed Weyl fermions $\chi_{1,2}$ and $\tilde{\chi}_{1,2}$ and four real scalar (or pseudoscalar) fields $\phi^{1,2,3,4}$,

$$\begin{aligned}\mathcal{L} \;=\; &\sum_{j=1,2} i\chi_j^\dagger \bar{\sigma}_\mu \partial_\mu \chi_j \;+\; \sum_{j=1,2} i\tilde{\chi}_j^\dagger \bar{\sigma}_\mu \partial_\mu \tilde{\chi}_j \;+\; \sum_{a=1,2,3,4} \tfrac{1}{2}(\partial_\mu \phi^a)^2 \\ &-\; V(\phi) \;-\; g\sum_{ij} \Phi_{ij} \times \tilde{\chi}_i^\top \sigma_2 \chi_j \;-\; g\sum_{ij} \Phi_{ij}^* \times \chi_j^\dagger \sigma_2 \tilde{\chi}_i^* \,,\end{aligned} \tag{1}$$

where

$$V(\phi) \;=\; \frac{\lambda}{8}\left(\sum_a (\phi^a)^2\right)^2 \;+\; \frac{m^2}{2}\sum_a (\phi^a)^2 \;+\; \text{const} \tag{2}$$

and

$$\Phi_{ij} \;=\; \phi^4 \times \delta_{ij} \;+\; i\sum_{a=1,2,3} \phi^a \times \tau_{ij}^a \;=\; \begin{pmatrix} \phi^4 + i\phi^3 & -\phi^2 + i\phi^1 \\ +\phi^2 + i\phi^1 & \phi^4 - i\phi^3 \end{pmatrix}. \tag{3}$$

Note: Φ is an $SU(2)$ matrix multiplied by a real number.

(a) The theory has $SU(2)_L \times SU(2)_R \times U(1)_V$ group of global symmetries. Show how all these symmetries act on the fermionic and scalar fields.

From now on, let $m^2 < 0$ so that $\langle \phi^a \rangle = v \times \delta^{a4} \neq 0$ and the symmetry group is spontaneously broken down to $SU(2)_V \times U(1)_V$.

(b) Describe the particle spectrum of the spontaneously broken theory: list all the particle species, their masses, and their quantum numbers with respect to the unbroken $SU(2)_V \times U(1)_V$ symmetries (*i.e.*, which $SU(2)_V$ multiplets they belong to, and what are their $U(1)_V$ charges).

(c) Assemble the Weyl spinor fields into 2 Dirac spinors Ψ_1 and Ψ_2, and re-write the Lagrangian (1) in terms of $\Psi_{1,2}$, $\sigma = \phi^4 - v$, and $\pi^{1,2,3} = \phi^{1,2,3}$.

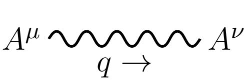
(d) Describe how the discrete symmetries P, C, and CP act on all the fields. For the fermions, give two descriptions, one in terms of the Dirac fermions Ψ_i and the other in terms of the Weyl fermions χ_i and $\tilde{\chi}_i$.

Also describe the action of $G = C \times e^{i\pi T^2}$ — which combines the charge conjugation with a 180° isospin rotation — and GP. For hadrons, one usually uses G instead of C because G commutes with the isospin.

2. Now consider Quantum Electro-Dynamics of photons γ, electrons $e^\mp$, and also charged scalar particles $S^\mp$. The Lagrangian of this theory is

$$\mathcal{L} \;=\; -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \overline{\Psi}(i\not{D} - m)\Psi + \; D_\mu\Phi^* D^\mu\Phi \;-\; M^2\Phi^*\Phi \;=\; \mathcal{L}^{\text{free}} + \mathcal{L}^{\text{interactions}}. \quad (4)$$

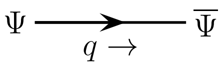
In the Feynman rules, the propagators and the external lines follow from the $\mathcal{L}^{\text{free}}$ while the vertices follow from the $\mathcal{L}^{\text{interactions}}$. Altogether, the Feynman rules for QED with both electrons and charged scalars are

Photonic propagator: 

$$= \frac{-ig^{\mu\nu}}{q^2 + i0}, \quad (F.1)$$

Incoming photon:  $= e_\mu(k, \lambda)$, (F.2)

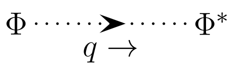
Outgoing photon:  $= e^*_\mu(k, \lambda)$, (F.3)

Electron propagator: 

$$= \frac{i}{\not{q} - m + i0}, \quad (F.4)$$

Incoming e^- or outgoing e^+:  $= u(p, s)$ or $v(p, s)$, (F.5)

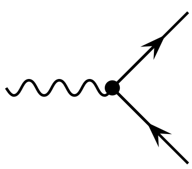
Outgoing e^- or incoming e^+:  $= \bar{u}(p, s)$ or $\bar{v}(p, s)$, (F.6)

Scalar propagator: 

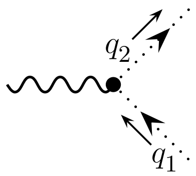
$$= \frac{i}{q^2 - M^2 + i0}, \quad (F.7)$$

Incoming S^- or outgoing S^+:  $= 1$, (F.8)

Outgoing S^- or incoming S^+:  $= 1$, (F.9)

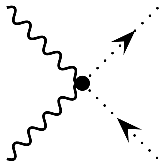
QED vertex $ee\gamma$:  $= +ie\gamma^\mu$, (F.10)

Scalar QED vertex $SS\gamma$:



$$= +ie(q_1 + q_2)^\mu \tag{F.11}$$

Seagull vertex $SS\gamma\gamma$:



$$= +2ie^2\, g^{\mu\nu}. \tag{F.12}$$

Note: the dotted lines (F.7–9) for the charged scalars have arrows. Also note that in the $S^- S^+ \gamma$ vertex (F.11), the directions of momenta q_1 and q_2 must agree with the arrows of the scalar lines; otherwise, the vertex becomes $+ie(q_1 - q_2)^\mu$ or $+ie(-q_1 + q_2)^\mu$ or $+ie(-q_1 - q_2)^\mu$.

(a) The QED Feynman rules (F.1–6) and (F.10) were explained in class. Explain the remaining rules (F.7–9) and (F.11–12) in terms of the Lagrangian (4). Note: don't re-derive the Feynman rules as such, just explain why the scalar lines and vertices are as in eqs. (F.7–9) and (F.11–12).

(b) Given the Feynman rules, draw the tree diagram(s) for the scalar pair production $e^- e^+ \to S^- S^+$ and calculate the tree-level amplitude $\langle S^-, S^+ \big| \mathcal{M} \big| e^-, e^+ \rangle$.
Hint: Mind the arrow directions on the dotted lines of scalars.

(c) Calculate the partial cross-section for the scalar pair production and compare its angular dependence with that of the $e^- e^+ \to \mu^- \mu^+$ process. Also, calculate the total cross-section $\sigma_{\rm tot}(e^- e^+ \to S^- S^+)$ and compare its energy dependence to that of $\sigma_{\rm tot}(e^- e^+ \to \mu^- \mu^+)$.
For simplicity, neglect the electron's mass m.

(d) Draw diagrams which contribute to the $\langle S^-, S^+ \big| \mathcal{M} \big| e^-, e^+ \rangle$ amplitude in the next-to-leading order of the perturbation theory. Don't evaluate their contributions, just draw a pageful of diagrams.

3. Now consider annihilation of the charged scalars into photons, $S^+S^- \to \gamma\gamma$.

   (a) Draw and evaluate **all** tree diagrams contributing to the $\langle\gamma\gamma|\,\mathcal{M}\,|S^+S^-\rangle$ amplitude. Make sure the amplitude respects the Bose symmetry between the two photons.

   (b) Write the tree amplitude as $\mathcal{M} = \mathcal{M}_{\mu\nu} \times e_1^{*\mu} e_2^{*\nu}$ and verify the Ward identities

$$k_1^\mu \times \mathcal{M}_{\mu\nu} \;=\; k_2^\nu \times \mathcal{M}_{\mu\nu} \;=\; 0. \tag{5}$$

   Hint: If these identities seem to be broken, go back to part (a) and make sure you have not missed a diagram. If this does not help, check your signs.

   (c) Sum $|\mathcal{M}|^2$ over the outgoing photon polarizations and calculate the partial cross-section of the $S^+S^- \to \gamma\gamma$ annihilation.

- For simplicity, assume $E \gg M$ and neglect the scalar mass M in your calculations.

⋆ Extra credit if you do take M into account, and do it right. But beware: the kinematics is much messier for $M \neq 0$, and you might need a few hours to work through the algebra.