

Please do not waste time and paper by copying the posted homework solutions or supplementary notes. If you need to use any homework result, simply reference the appropriate question or equation and go ahead. Likewise, don't re-derive anything I derived in class.

1. The first problem is about tree-level gluon scattering, $gg \rightarrow gg$.

(a) Draw all tree diagrams for this process. Use crossing symmetry to write the net amplitude as

$$\mathcal{M}(g_1^a, g_2^b, g_3^c, g_4^d) = G^{abcd} \times \mathcal{M}_s + G^{acdb} \times \mathcal{M}_t + G^{adbc} \times \mathcal{M}_u \quad (1)$$

where G^{abcd} , *etc.*, are group factors depending on the colors of the four gluons while the \mathcal{M}_s , \mathcal{M}_t , and \mathcal{M}_u amplitudes depend on their momenta and polarizations. Thanks to the crossing symmetry,

$$\mathcal{M}_s \equiv \mathcal{M}(1, 2, 3, 4), \quad \mathcal{M}_t \equiv \mathcal{M}(1, 3, 4, 2), \quad \mathcal{M}_u \equiv \mathcal{M}(1, 4, 2, 3), \quad (2)$$

for the same analytic function \mathcal{M} applied to 3 different ordering of the four gluons. (For simplicity, treat all 4 gluons as incoming, $k_1 + k_2 + k_3 + k_4 = 0$.)

(b) Show that group factor G^{abcd} has the same index symmetry as the Riemann tensor in gravity,

$$G^{abcd} = -G^{bacd} = -G^{abdc} = +G^{cdab}, \quad (3)$$

$$G^{abcd} + G^{acdb} + G^{adbc} = 0. \quad (4)$$

Eqs. (3) should be obvious (if they are not, you may have a wrong G^{abcd}), but eq. (4) takes some work. To prove it, use the identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

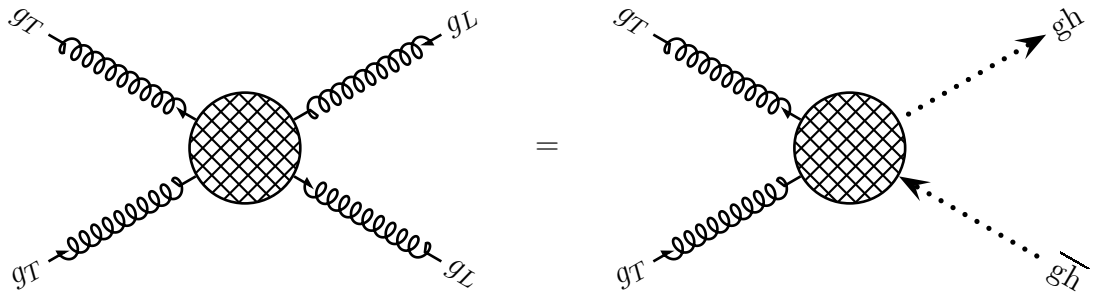
(c) Sum / average the 4-gluon |amplitude|² over all the colors and show that

$$\overline{|\mathcal{M}|^2} = \frac{C^2(G)}{2 \dim(G)} \times (3|\mathcal{M}_s|^2 + 3|\mathcal{M}_t|^2 + 3|\mathcal{M}_u|^2 - |\mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u|^2). \quad (5)$$

- (d) Prove the weak Ward identity for the 4-gluon amplitude (1): If one gluon has $e^\mu \propto k^\mu$ while the other three gluons are transverse, then $\mathcal{M} = 0$.

Hint: Show that in this case $\mathcal{M}_s = \mathcal{M}_t = \mathcal{M}_u$, then use eq. (4).

- (e) Now suppose only two gluons are transverse while the other two have unphysical polarizations (longitudinal or temporal). Or rather, let the two unphysical gluons have null polarization vectors $e^2 = 0$, specifically $e_3^\mu = k_3^\mu$ for one and $e_4^\mu k_{4\mu} = 1$ for the other. Show that in this case, the 4-gluon amplitude is exactly equal to the amplitude where the unphysical gluons are replaced with a ghost and an antighost,



Finally, let's calculate the amplitudes (2) and the partial cross-section for the four transverse gluons. For simplicity, work in the center-of-mass frame and use linear polarizations for each gluon, either \parallel to the plane of scattering or \perp to it. For the set of 4 gluons there are 16 choices of such polarizations, but the symmetries forbid some combinations and relate other combinations to each other.

- (f) Spell out which polarized $gg \rightarrow gg$ processes are forbidden and which are allowed. Write down the symmetry relations between the allowed processes. How many of them are independent?
- (g) Calculate the amplitudes (2) and the partial cross-section for the simplest choice of polarizations: all 4 gluons are \perp to the scattering plane.

★ Optional exercise, for extra credit:

Calculate the partial cross-sections for the other independent polarizations.

Warning: such amplitudes involve much messier algebra than the all- \perp case (g), so use Mathematica or calculate them numerically as functions of the scattering angle θ .

If you try to calculate them by hand, you are liable to make more algebraic mistakes than you can fix during the time available for this exam.

2. The second problem is about QCD in $2 + 1$ spacetime dimensions. Three exams ago ([Fall 2008 midterm](#), problem 3) you have analyzed the *topologically massive Yang–Mills theory* in 3D

$$\mathcal{L} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{CS}} = -\frac{1}{2g^2} \text{tr}(\mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}) + \frac{k}{4\pi} \epsilon^{\lambda\mu\nu} \text{tr} \left(\mathcal{A}_\lambda \partial_\mu \mathcal{A}_\nu + \frac{2i}{3} \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu \right) \quad (6)$$

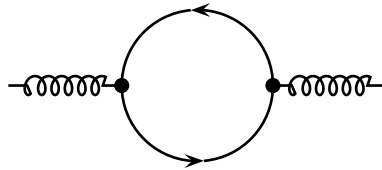
and saw that for integer *Chern–Simons level* k , the action is gauge invariant (modulo 2π) but the gluons are massive (for $k \neq 0$). In this exercise, we shall *generate* the Chern–Simons term by integrating out the massive quarks.

Let’s start with 3D $SU(N)$ gauge theory with a single fundamental multiplet \mathbf{N} of quarks (*i.e.*, N colors, one flavor),

$$\mathcal{L} = \frac{-1}{2g^2} \text{tr}(\mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu}) + \bar{\Psi}(i\not{D} - m)\Psi. \quad (7)$$

Note that in odd spacetime dimensions, massive Dirac fermions have no Parity symmetry even at the tree level. At higher orders of perturbation theory, the quark loop diagrams yield parity-violating amplitudes thanks to $\text{tr}(\gamma^\lambda \gamma^\mu \gamma^\nu) = 2i\epsilon^{\lambda\mu\nu}$ in 3D (and similar formulæ in higher odd dimensions), so the gluon sector of the theory also becomes parity-violating.

- (a) Evaluate the one loop diagram



and show that for *small* gluon momentum $|p| \ll m$ it yields

$$\Sigma_{\psi \text{ loop}}^{\mu a, \nu b}(p) = \frac{g^2 \delta^{ab}}{8\pi} \left(-ip_\lambda \epsilon^{\lambda\mu\nu} + \frac{p^\mu p^\nu - g^{\mu\nu} p^2}{3m} + O\left(\frac{p^3}{m^2}\right) \right). \quad (8)$$

- (b) Similarly, show that for three external gluons with small momenta (compared to the fermion's mass m), the one-loop amplitude is

$$iV_{\lambda\mu\nu}^{abc} = \text{[Diagram 1]} + \text{[Diagram 2]} = \frac{ig^3}{8\pi} f^{abc} \epsilon_{\lambda\mu\nu} + O\left(\frac{p}{m}\right). \quad (9)$$

- (c) Show that for quark loops with four or more external gluons with small momenta, all the one-quark-loop amplitudes are suppressed by negative powers of the quark mass m .

Now consider the Functional Integral for the $d = 3$ QCD. Let us integrate $\iint D[\Psi(x)] \iint D[\bar{\Psi}(x)]$ over the quark fields for fixed gauge fields $A_\mu^a(x)$. The result of this integration is an effective quantum theory of the gauge fields with action

$$S[A_\mu^a] = S_{YM}[A_\mu^a] - i \log \text{Det}(i\mathcal{D} - m). \quad (10)$$

- (d) Use the results of questions (a), (b) and (c) to show that in the large quark mass m limit,

$$-i \log \text{Det}(i\mathcal{D} - m) = \frac{1}{2} \int d^3x \left\{ \frac{1}{8\pi} \epsilon^{\lambda\mu\nu} \text{tr}(\mathcal{A}_\lambda \mathcal{F}_{\mu\nu} - \frac{2i}{3} \mathcal{A}_\lambda \mathcal{A}_\mu \mathcal{A}_\nu) + O\left(\frac{1}{m}\right) \right\} \quad (11)$$

and consequently, the effective low-energy quantum theory is precisely the topologically massive Yang–Mills theory (6) with Chern–Simons level $k = \frac{1}{2}$.

Note: fractional Chern-Simons levels k break gauge invariance of the functional integral. To get an integer k , we need to integrate out an even number of quark flavors.

- (e) Now let's have several flavors of massive quarks, some with $m_f > 0$ and some with $m_f < 0$ (in 3D, this makes a difference). Show that when we integrate out all these quarks, we end up with the Chern–Simons level

$$k = \frac{\#(m_f > 0) - \#(m_f < 0)}{2}. \quad (12)$$

Note: 3D QCD with an odd number of flavors is inconsistent without a *tree-level* Chern-Simons term with a half-integral coefficient $k_0 \in \mathbf{Z} + \frac{1}{2}$. When you integrate out the quarks, the *net* CS level $k = k_0 + \frac{1}{2}\#(m_f > 0) - \frac{1}{2}\#(m_f < 0)$ becomes an integer.