

# QED Feynman rules

Quantum ElectroDynamics of QED is the theory of EM field  $A_\mu$  coupled to the electron field  $\Psi$  (and optionally other charged fermion fields). The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\mathcal{D} - m)\Psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\cancel{\partial} - m)\Psi + eA_\mu \times \bar{\Psi}\gamma^\mu\Psi\end{aligned}\tag{1}$$

where the first 2 terms on the last line describe free photons and electrons  $e^\pm$ , and the third term is treated as a perturbation.

The two different field types give rise to two different propagators (internal lines) in QED Feynman rules. An electron propagator is drawn as a solid line with an arrow indicating which end of the line belongs to  $\Psi$  field and which to  $\bar{\Psi}$ ,

$$\bar{\Psi}_\alpha \bullet \xleftarrow{\leftarrow q} \bullet \Psi_\beta = \left[ \frac{i}{\cancel{q} - m + i0} \right]_{\alpha\beta}.\tag{2}$$

The smaller arrow near  $q$  indicates the direction of the momentum flow. Both arrows should have the same direction; otherwise we would have

$$\bar{\Psi}_\alpha \bullet \xrightarrow{q \rightarrow} \bullet \Psi_\beta = \left[ \frac{i}{-\cancel{q} - m + i0} \right]_{\alpha\beta}.\tag{3}$$

The photon propagator is drawn as a wavy line without arrow,

$$A^\mu \bullet \text{~~~~~} \bullet A^\nu = \frac{iC^{\mu\nu}(q)}{q^2 + i0}\tag{4}$$

where

$$C^{\mu\nu}(q) = -g^{\mu\nu} + q^\mu \times t^\nu(q) + q^\nu t^\mu(q)\tag{5}$$

and the  $t^\mu(q)$  vectors depend on the gauge condition for the EM fields (*cf.* homework 8). When the photon is coupled to conserved electric currents, the  $q^\mu t^\nu + t^\mu q^\nu$  terms do not contribute because  $q^\mu \times J_\mu(q) = 0$ . Consequently, all physical QED amplitudes turn out to be the same in all gauges, provided one uses the same gauge for all the propagators in all Feynman diagrams contributing to the same process.

In this class we shall use the *Feynman gauge* where  $t^\nu \equiv 0$  and the propagator is simply

$$A^\mu \bullet \underset{q \rightarrow}{\text{wavy}} \bullet A^\nu = \frac{-ig^{\mu\nu}}{q^2 + i0}. \quad (6)$$

Defining the Feynman gauge in terms of restrictions on the  $A^\mu(x)$  fields is rather complicated, so I'll postpone this issue until April; all we need for now is the photon propagator (6).

Another commonly used gauge is the Landau gauge in which the  $A^\mu(x)$  field satisfies a Lorentz-invariant condition  $\partial_\mu A^\mu(x) \equiv 0$ . In the Landau gauge, the photon propagator is

$$A^\mu \bullet \underset{q \rightarrow}{\text{wavy}} \bullet A^\nu = \frac{-i}{q^2 + i0} \times \left( g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2 + i0} \right). \quad (7)$$

QED vertices follow from electron-photon interaction term  $eA_\mu \times \bar{\Psi}\gamma^\mu\Psi$ . There is only one vertex type, namely

$$\begin{array}{c} \beta \\ \nearrow \\ \mu \text{ wavy} \bullet \\ \searrow \\ \alpha \end{array} = (+ie\gamma^\mu)_{\beta\alpha}. \quad (8)$$

This vertex has valence = 3 and the 3 lines must be of specific types: one wavy (photonic) line, one solid line with incoming arrow, and one solid line with outgoing arrow.

Now consider the external lines. The momentum-space Feynman rules of the scalar theory do not have any factors due to external lines, but QED Feynman rules are more complicated. The photonic external lines carry *polarization vectors*:

$$\underset{k \rightarrow}{\text{wavy}} \bullet = e_\mu(k, \lambda) \quad (9)$$

for an incoming photon, and

$$\bullet \underset{k \rightarrow}{\text{wavy}} = e_\mu^*(k, \lambda) \quad (10)$$

for an outgoing photon.

The fermionic external lines carry plane-wave Dirac spinors  $u(p, s)$ ,  $v(p, s)$ ,  $\bar{u}(p, s)$ , and  $\bar{v}(p, s)$ . Specifically,

$$\text{an incoming electron } e^- \text{ carries } \underset{p \rightarrow}{\text{solid}} \bullet = u_\alpha(p, s), \quad (11)$$

an outgoing electron  $e^-$  carries  $\bullet \xrightarrow{p \rightarrow} = \bar{u}_\alpha(p, s),$  (12)

an incoming positron  $e^+$  carries  $\xleftarrow{p \rightarrow} \bullet = \bar{v}_\alpha(p, s),$  (13)

an outgoing positron  $e^+$  carries  $\bullet \xleftarrow{p \rightarrow} = v_\alpha(p, s).$  (14)

Note that for positrons, the direction of the arrow is opposite from the particle's (and its momentum): An incoming positron has an outgoing line (but in-flowing momentum) while an outgoing positron has an incoming line (but an outflowing momentum). For the electrons, the line has the same directions as the particle incoming for an incoming  $e^-$  and outgoing for an outgoing  $e^-$ . In general, the arrows in fermionic lines follow the flow of the electric charge (in units of  $-e$ ), hence opposite directions for electrons and positrons.

The QED vertex (8) has one incoming fermionic line and one outgoing, and we may think of them as being two segments of single continuous line going through the vertex. From this point of view, a fermionic line enters a diagram as an incoming  $e^-$  or an outgoing  $e^+$ , goes through a sequence of vertices and propagators, and eventually exits the diagram as an outgoing  $e^-$  or an incoming  $e^+$ ,

Diagram (15) shows a horizontal fermionic line with arrows pointing to the right. It starts on the left with a bracket labeled  $e^-$  in and  $e^+$  out. The line passes through four vertices, each marked with a black dot. From each vertex, a wavy line (representing a photon) extends upwards. The line ends on the right with a bracket labeled  $e^-$  out and  $e^+$  in. An ellipsis between the second and third vertices indicates a continuation of the line. (15)

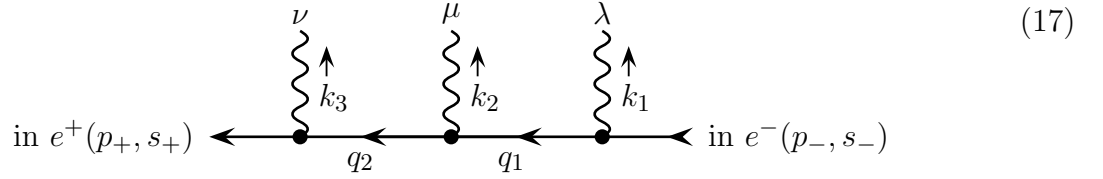
(The photonic lines here may be external or internal; if internal, they connect to some other fermionic lines, or maybe even to the same line at another vertex.) Alternatively, a fermionic line may form a closed loop, like

Diagram (16) shows a square fermionic loop with arrows indicating a clockwise direction. Each of the four vertices is marked with a black dot. From each vertex, a wavy line (representing a photon) extends outwards. (16)

The continuous fermionic lines such as (15) or (16) are convenient for handling the Dirac indices of vertices, fermionic propagators, and external lines. For an open line such as (15),

the rule is to read the line in order, from its beginning to its end, spell all the vertices, the propagators, and the external line factors in the same order *right-to-left*, then multiply them together as Dirac matrices.

For example, consider a diagram where an incoming electron and incoming positron annihilate into 3 photons, real or virtual. This diagram has a fermionic line which starts at the incoming  $e^-$ , goes through 3 vertices and 2 propagators, and exits at the incoming  $e^+$  as shown below:



The propagators here carry momenta  $q_1 = p_- - k_1$  and  $q_2 = q_1 - k_2 = k_3 - p_+$ . The fermionic line (17) carries the following factors:

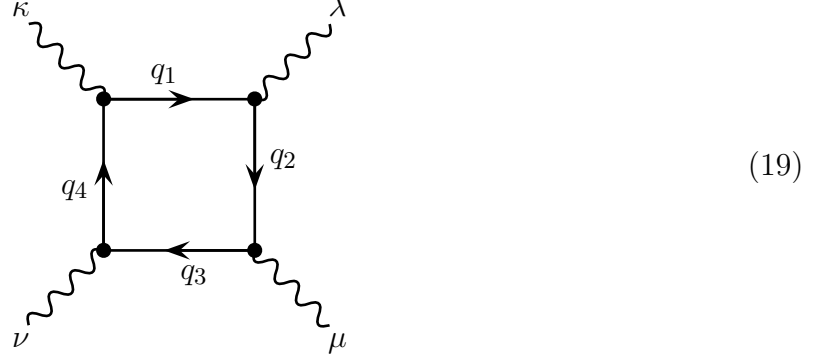
- $u(p_-, s_-)$  for the incoming  $e^-$ ;
- $+ie\gamma^\lambda$  for the first vertex (from the right);
- $\frac{i}{\not{q}_1 - m + i\epsilon}$  for the first propagator;
- $+ie\gamma^\mu$  for the second vertex;
- $\frac{i}{\not{q}_2 - m + i\epsilon}$  for the second propagator;
- $+ie\gamma^\nu$  for the third vertex;
- $\bar{v}(p_+, s_+)$  for the incoming  $e^+$ .

Reading all these factors in the order of the line (17), tail-to-head, and multiplying them right-to-left, we get the following Dirac ‘sandwich’

$$\bar{v}(p_+, s_+) \times (+e\gamma^\nu) \times \frac{i}{\not{q}_2 - m + i\epsilon} \times (+ie\gamma^\mu) \times \frac{i}{\not{q}_1 - m + i\epsilon} \times (+ie\gamma^\lambda) \times u(p_-, s_-). \quad (18)$$

In this formula, all the Dirac indices are suppressed; the rule is to multiply all factors as Dirac matrices (and row / column spinors) *in this order*.

For a closed fermionic loop such as (16), the rule is to start at an arbitrary vertex or propagator, follow the line until one gets back to the starting point, multiply all the vertices and the propagators *right-to-left in the order of the line*, then take the trace of the matrix product. For example, the loop

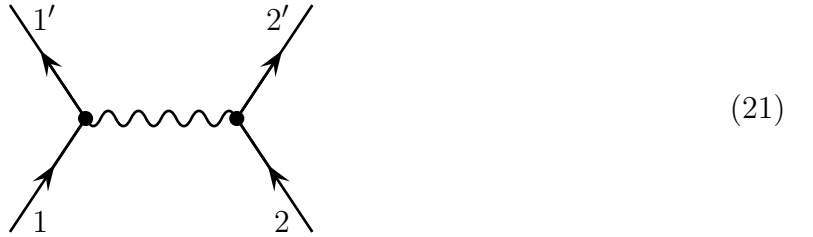


produces Dirac trace

$$\begin{aligned} \text{tr} \left[ (+ie\gamma^\kappa) \times \frac{i}{\not{q}_4 - m + i\epsilon} \times (+ie\gamma^\nu) \times \frac{i}{\not{q}_3 - m + i\epsilon} \times \right. \\ \left. \times (+ie\gamma^\mu) \times \frac{i}{\not{q}_2 - m + i\epsilon} \times (+ie\gamma^\lambda) \times \frac{i}{\not{q}_1 - m + i\epsilon} \right]. \end{aligned} \quad (20)$$

Note that a trace of a matrix product depends only on the cyclic order of the matrices, ( $\text{tr}(ABC \cdots YZ) = \text{tr}(BC \cdots YZA) = \text{tr}(C \cdots YZAB) = \cdots = \text{tr}(ZABC \cdots Y)$ ). Thus, in eq. (20), we may start the product with any vertex or propagator — as long as we multiply them all in the correct *cyclic order*, the trace will be the same.

As to the Lorentz vector indices  $\lambda, \mu, \nu, \dots$ , the index of a vertex should be contracted to the index of the photonic line connected to that vertex. For example, the following diagram for  $e^- + e^- \rightarrow e^- + e^-$  scattering



evaluates to

$$i\mathcal{M} = \left( \bar{u}(p'_1, s'_1) \times (+ie\gamma_\mu) \times u(p_1, s_1) \right) \times \left( \bar{u}(p'_2, s'_2) \times (+ie\gamma_\nu) \times u(p_2, s_2) \right) \times \frac{-ig^{\mu\nu}}{q^2}. \quad (22)$$

Here we have used the Feynman gauge for the photon propagator, but any other gauge would produce exactly the same amplitude

$$\begin{aligned}
i\mathcal{M} &= \bar{u}'_1(ie\gamma_\mu)u_1 \times \bar{u}'_2(ie\gamma_\nu)u_2 \times \frac{i(-g^{\mu\nu} + t^\mu q^\nu + q^\mu t^\nu)}{q^2} \\
&= \bar{u}'_1(ie\gamma_\mu)u_1 \times \bar{u}'_2(ie\gamma_\nu)u_2 \times \frac{-ig^{\mu\nu}}{q^2}
\end{aligned} \tag{23}$$

because

$$\bar{u}'_1(ie\gamma_\mu)u_1 \times q^\mu = \bar{u}'_2(ie\gamma_\nu)u_2 \times q^\nu = 0. \tag{24}$$

To prove this formula, we note that the spinors  $u_1 \equiv u(p_1, s_1)$  and  $\bar{u}'_1 \equiv \bar{u}(p'_1, s'_1)$  satisfy Dirac equation

$$\not{p}_1 u_1 = m u_1, \quad \bar{u}'_1 \not{p}'_1 = m \bar{u}'_1. \tag{25}$$

Moreover,  $q = p'_1 - p_1$  and hence

$$\bar{u}'_1 \gamma_\mu u_1 \times q^\mu = \bar{u}'_1 \not{q} u_1 = \bar{u}'_1 (\not{p}'_1 - \not{p}_1) u_1 = (m \bar{u}'_1) u_1 - \bar{u}'_1 (m u_1) = 0. \tag{26}$$

Similarly,  $q = p_2 - p'_2$  and hence

$$\bar{u}'_2 \gamma_\mu u_2 \times q^\mu = \bar{u}'_2 \not{q} u_2 = \bar{u}'_2 (\not{p}_2 - \not{p}'_2) u_2 = \bar{u}'_2 (m u_2) u_2 - (m \bar{u}'_2) = 0. \tag{27}$$

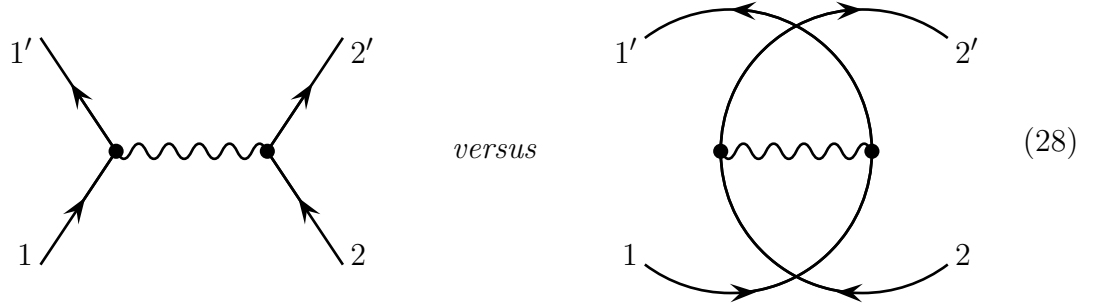
In general, an individual Feynman diagram is not always gauge-independent. However, when one sums over all diagrams contributing to some scattering process at some order, the sum is always gauge invariant. We shall return to this issue later this semester.

To complete the QED Feynman rules, we need to keep track of the ‘−’ signs arising from re-ordering of fermionic fields and creation / annihilation operators. To save time, I will not go through the gory details of the perturbation theory. Instead, let me simply state the rules for the overall sign of a Feynman diagram in terms of the continuous fermionic lines:

- There is a ‘−’ sign for every closed fermionic loop.
- There is a ‘−’ sign for every open fermionic line which begins at an outgoing positron and ends at an incoming positron.

- There is a ‘ $-$ ’ sign for every crossing of the fermionic lines.

Although the number of such crossing depends on how we draw the diagram on a 2D sheet of paper, for example



However,  $\#\text{crossings} \bmod 2$  is a topological invariant, and that’s all we need to determine the overall sign of the diagram.

- ★ If multiple Feynman diagrams contribute to the same process, all the diagram should have external legs sticking out the diagram in the same order for all the diagrams. Or at least all the fermionic external legs should stick out in the same order, which should also agree with the order of fermions in the bra and ket states of the S–matrix element  $\langle e^{-'}, \dots, e^{-'}, e^{+'}, \dots, e^{+'}, \gamma', \dots, \gamma' | \mathcal{M} | e^{-}, \dots, e^{-}, e^{+}, \dots, e^{+}, \gamma, \dots, \gamma \rangle$  for the process in question.

Finally, QED is usually extended to include other charged fermions besides  $e^{\mp}$ . The simplest extension includes the muons  $\mu^{\mp}$  and the tau leptons  $\tau^{\mp}$  which behave exactly like the electrons, except for larger masses: while  $m_e = 0.51100$  MeV,  $m_{\mu} = 105.66$  MeV and  $m_{\tau} = 1777$  MeV. In terms of the Feynman rules, the muons and the taus have exactly the same vertices, propagators, or external line as the electrons, except for a different mass  $m$  in the propagators. To distinguish between the 3 lepton species, one should label the solid lines with  $e$ ,  $\mu$ , or  $\tau$ . Different species do not mix, so a label belongs to the whole continuous fermionic line; for an open line, the species must agree with the incoming / outgoing particles at the ends of the line; for a closed loop, one should sum over the species  $\ell = e, \mu, \tau$ .