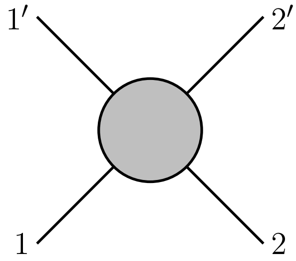
Mandelstam Variables

Consider any kind of a 2 particles → 2 particles process



(1)

All Lorentz invariant combinations of the 4 incoming or outgoing momenta p_1^μ, p_2^μ, $p_1'^\mu$, and $p_2'^\mu$ may be expressed in terms of the 3 Mandelstam's variables

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_1'+p_2')^2, \\ t &= (p_1-p_1')^2 = (p_2'-p_2)^2, \\ u &= (p_1-p_2')^2 = (p_1'-p_2)^2. \end{aligned} \tag{2}$$

Moreover, only 2 out of these 3 variables are independent because for the on-shell momenta

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 . \tag{3}$$

In terms of Mandelstam's variables, Lorentz products of momenta are given by

$$\begin{aligned} 2(p_1p_2) &= s - m_1^2 - m_2^2 , \\ 2(p_1'p_2') &= s - m_1'^2 - m_2'^2 , \\ 2(p_1p_1') &= m_1^2 + m_1'^2 - t, \\ 2(p_2p_2') &= m_2^2 + m_2'^2 - t, \\ 2(p_1p_2') &= m_1^2 + m_2'^2 - u, \\ 2(p_2p_1') &= m_2^2 + m_1'^2 - u. \end{aligned} \tag{4}$$

In particular, for the $e^-e^+ \to \gamma\gamma$ annihilation process $p_- + p_+ \to k_1 + k_2$,

$$\begin{aligned} s + t + u &= 2m_e^2 , \\ 2(p_-p_+) &= s - 2m_e^2 , \\ 2(k_1k_2) &= s, \\ 2(p_-k_1) = 2(p_+k_2) &= m_e^2 - t, \\ 2(p_-k_2) = 2(p_+k_1) &= m_e^2 - u. \end{aligned} \tag{5}$$

Proof of eq. (2):

$$\begin{aligned} s \;+\; t \;+\; u \;&=\; (p_1+p_2)^2 \;+\; (p_1-p_1')^2 \;+\; (p_1-p_2')^2 \\ &=\; 3p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 \;+\; 2(p_1p_2) \;-\; 2(p_1p_1') \;-\; 2(p_1p_2') \\ &=\; p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 +\; 2p_1 \times (p_1+p_2-p_1'-p_2'=0) \\ &=\; p_1^2 \;+\; p_2^2 \;+\; p_1'^2 \;+\; p_2'^2 \\ &=\; m_1^2 \;+\; m_2^2 \;+\; m_1'^2 \;+\; m_2'^2\,. \end{aligned} \tag{6}$$

Proofs of eqs. (4):

$$\begin{aligned} 2(p_1p_2) \;&=\; (p_1+p_2)^2 \;-\; p_1^2 \;-\; p_2^2 \;=\; s \;-\; m_1^2 \;-\; m_2^2\,, \\ 2(p_1'p_2') \;&=\; (p_1'+p_2')^2 - p_1'^2 - p_2'^2 \;=\; s \;-\; m_1'^2 \;-\; m_2'^2\,, \\ 2(p_1p_1') \;&=\; p_1^2 + p_1'^2 \;-\; (p_1-p_1')^2 \;=\; m_1^2 + m_1'^2 \;-\; t, \\ 2(p_2p_2') \;&=\; p_2^2 + p_2'^2 \;-\; (p_2-p_2')^2 \;=\; m_2^2 + m_2'^2 \;-\; t, \\ 2(p_1p_2') \;&=\; p_1^2 + p_2'^2 \;-\; (p_1-p_2')^2 \;=\; m_1^2 + m_2'^2 \;-\; u, \\ 2(p_2p_1') \;&=\; p_2^2 + p_1'^2 \;-\; (p_2-p_1')^2 \;=\; m_2^2 + m_1'^2 \;-\; u. \end{aligned} \tag{7}$$