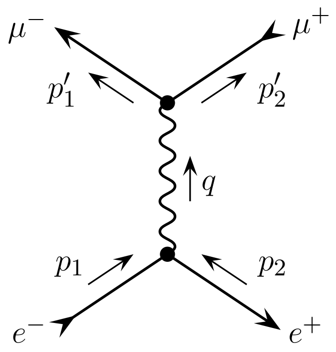
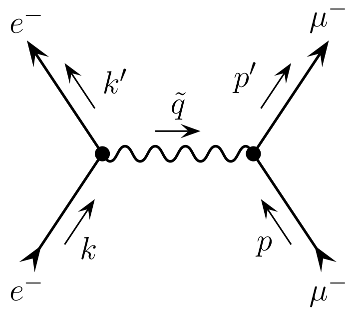
# Crossing Symmetry

Consider 2 processes in QED: (a) muon pair production $E^- e^+ \to \mu^- \mu^+$ and (b) elastic electron-muon scattering $e^- \mu^- \to e^- \mu^-$. There is one tree diagram for each process,




(1)

and they look very similar. Topologically, the two diagrams are identical. The only difference is in the physical meaning of the external legs and the signs of the external momenta (incoming versus outgoing). Here is the correspondence table:

| **pair production** | **scattering** | **momenta** |
|---|---|---|
| incoming $e^-$ | incoming $e^-$ | $k \leftrightarrow +p_1$ |
| incoming $e^+$ | outgoing $e^-$ | $k' \leftrightarrow -p_2$ |
| outgoing $\mu^-$ | outgoing $\mu^-$ | $p' \leftrightarrow +p_1'$ |
| outgoing $\mu^+$ | incoming $\mu^-$ | $p \leftrightarrow -p_2'$ |

(2)

Now let's compare the tree-level amplitudes: For the pair production,

$$\mathcal{M}^{\text{pair}} = \frac{e^2}{s} \times \bar{v}(e^+)\gamma^\nu u(e^-) \times \bar{u}(\mu^-)\gamma_\nu v(\mu^+) \tag{3}$$

where $s = (p_1 + p_2)^2$, and after summing over all the spins we get

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{pair}}\right|^2 = \frac{e^4}{s^2} \times \text{tr}\Big((\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\lambda\Big) \times \text{tr}\Big((\not{p}_1' + M_\mu)\gamma_\nu(\not{p}_2' - M_\mu)\gamma_\lambda\Big). \tag{4}$$

For the scattering,

$$\mathcal{M}^{\text{scatt}} = \frac{e^2}{t} \times \bar{u}(e^{-\prime})\gamma^\nu u(e^-) \times \bar{u}(\mu^{-\prime})\gamma_\nu u(\mu^-) \tag{5}$$

where $t = (k - k')^2$, and after summing over all the spins we get

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{scatt}}\right|^2 = \frac{e^4}{t^2} \times \text{tr}\Big((\not{k}' + m_e)\gamma^\nu(\not{k} + m_e)\gamma^\lambda\Big) \times \text{tr}\Big((\not{p}' + M_\mu)\gamma_\nu(\not{p} + M_\mu)\gamma_\lambda\Big). \tag{6}$$

The right hand sides of eqs. (4) and (6) are exactly the same analytic functions of the momenta, provided we identify the momenta in the two processes according to the table (2),

$$k \leftrightarrow +p_1\,, \quad k' \leftrightarrow -p_2\,, \quad p \leftrightarrow -p_2'\,, \quad p' \leftrightarrow +p_1'\,. \tag{7}$$

Indeed, under this mapping,

$$\begin{aligned} t^{\text{scatt}} = (k - k')^2 \;&\leftrightarrow\; s^{\text{pair}} = (p_1 + p_2)^2, \\ \text{tr}\Big((\not{k}' + m_e)\gamma^\nu(\not{k} + m_e)\gamma^\lambda\Big)^{\text{scatt}} \;&\leftrightarrow\; -\,\text{tr}\Big((\not{p}_2 - m_e)\gamma^\nu(\not{p}_1 + m_e)\gamma^\lambda\Big)^{\text{pair}}, \\ \text{tr}\Big((\not{p}' + M_\mu)\gamma_\nu(\not{p} + M_\mu)\gamma_\lambda\Big)^{\text{scatt}} \;&\leftrightarrow\; -\,\text{tr}\Big((\not{p}_1' + M_\mu)\gamma_\nu(\not{p}_2' - M_\mu)\gamma_\lambda\Big)^{\text{pair}}, \end{aligned} \tag{8}$$

and hence

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{scatt}}\right|^2 \leftrightarrow \sum_{\text{spins}} \left|\mathcal{M}^{\text{pair}}\right|^2. \tag{9}$$

To be precise, the correspondence in eq. (9) involves analytic continuation rather than outright equality because positive particle energies in scattering map onto negative energies in pair production and vice verse. Thus,

$$\sum_{\text{spins}} \left|\mathcal{M}^{\text{pair}}\right|^2 = F(p_1, p_2, p_1', p_2') \quad \text{and} \quad \sum_{\text{spins}} \left|\mathcal{M}^{\text{scatt}}\right|^2 = F(k, -k', p', -p) \tag{10}$$

*for the same analytic function $F$ of the momenta,* but for the pair production this function is evaluated for $p_2^0 > 0$ and $p_2'^0 > 0$, while for the scattering we use it for $p_2^0 = -k'^0 < 0$ and $p_2'^0 = -p^0 < 0$.

Relations such as (9) between processes described by similar Feynman diagrams (but with different identifications of the external legs as incoming or outgoing) are called *crossing symmetries.* And such crossing symmetries apply to amplitudes themselves and not just spin-summed $|\mathcal{M}|^2$, provided one properly maps the spin states of incoming and outgoing particles onto each other. For example, in the ultra-relativistic limit of muon pair production, the polarized amplitudes (for particles of definite helicities) are

$$\begin{aligned}
\left\langle \mu_L^-, \mu_R^+ \right| \mathcal{M} \left| e_L^-, e_R^+ \right\rangle \;&=\; \left\langle \mu_R^-, \mu_L^+ \right| \mathcal{M} \left| e_R^-, e_L^+ \right\rangle \;=\; 2e^2 \times \frac{u^{\rm pair}}{s^{\rm pair}}\,, \\
\left\langle \mu_R^-, \mu_L^+ \right| \mathcal{M} \left| e_L^-, e_R^+ \right\rangle \;&=\; \left\langle \mu_L^-, \mu_R^+ \right| \mathcal{M} \left| e_R^-, e_L^+ \right\rangle \;=\; 2e^2 \times \frac{t^{\rm pair}}{s^{\rm pair}}\,, \\
\text{for all other sets of helicities,} \quad \left\langle \mu^-, \mu^+ \right| \mathcal{M} \left| e^-, e^+ \right\rangle \;&=\; 0.
\end{aligned} \tag{11}$$

In the similar ultra-relativistic limit of the electron-muon scattering, the polarized amplitudes are

$$\begin{aligned}
\left\langle e_L^-, \mu_L^- \right| \mathcal{M} \left| e_L^-, \mu_L^- \right\rangle \;&=\; \left\langle e_R^-, \mu_R^- \right| \mathcal{M} \left| e_R^-, \mu_R^- \right\rangle \;=\; 2e^2 \times \frac{s^{\rm scatt}}{t^{\rm scatt}} \times e^{i\,\rm phase}, \\
\left\langle e_R^-, \mu_R^- \right| \mathcal{M} \left| e_L^-, \mu_L^- \right\rangle \;&=\; \left\langle e_L^-, \mu_L^- \right| \mathcal{M} \left| e_R^-, \mu_R^- \right\rangle \;=\; 2e^2 \times \frac{u^{\rm scatt}}{t^{\rm scatt}} \times e^{i\,\rm phase}, \\
\text{for all other sets of helicities,} \quad \left\langle e^-, \mu^- \right| \mathcal{M} \left| e^-, \mu^- \right\rangle \;&=\; 0.
\end{aligned} \tag{12}$$

Up to overall phases, the scattering amplitudes look exactly line the pair production amplitudes, provided we cross the Mandelstam variables $(s, t, u)$ according to eqs. (7),

$$\begin{aligned}
s^{\rm pair} &= (p_1 + p_2)^2 = (p_1' + p_2')^2 \leftrightarrow t^{\rm scatt} = (k - k')^2 = (p' - p)^2, \\
t^{\rm pair} &= (p_1 - p_1')^2 = (p_2' - p_2)^2 \leftrightarrow u^{\rm scatt} = (p' - k')^2 = (k - p)^2, \\
u^{\rm pair} &= (p_1 - p_2')^2 = (p_1' - p_2)^2 \leftrightarrow s^{\rm scatt} = (p' + k')^2 = (p + k)^2,
\end{aligned} \tag{13}$$

and also cross the helicities as

$$\lambda(e^{-\prime}) \;=\; -\lambda(e^+), \quad \lambda(\mu^-) \;=\; -\lambda(\mu^{+\prime}). \tag{14}$$

In other words,

$$\begin{aligned}
&\mathcal{M}^{\rm pair}\; f(s = s^{\rm pair}, t = t^{\rm pair}, u = u^{\rm pair}; \text{helicities}), \\
&\mathcal{M}^{\rm scatt}\; f(s = t^{\rm scatt}, t = u^{\rm scatt}, u = s^{\rm scatt}; \text{crossed helicities}) \times e^{i\,\rm phase},
\end{aligned} \tag{15}$$

*for exactly the same analytic finction* $f(s, t, u)$. However, the scattering and the pair production use this function for different domains of $s, t, u$: for the pair production $s > 0$ while $t, u < 0$, but for the scattering $u > 0$ while $s, t < 0$.