Dimensional Analysis and Allowed QFT Couplings

In $\hbar = c = 1$ units, all quantities are measured in units of energy to some power. For example $\big[m\big] = \big[p^\mu\big] = E^{+1}$ while $\big[x^\mu\big] = E^{-1}$ where $[m]$ stands for the *dimensionality* of the mass rather than the mass itself, and ditto for the $\big[p^\mu\big]$, $\big[x^\mu\big]$, *etc.* The action

$$S = \int d^4x\,\mathcal{L}$$

is dimensionless (in $\hbar \neq 1$ units, $\big[S\big] = \hbar$), so the Lagrangian of a 4D field theory has dimensionality $[\mathcal{L}] = E^{+4}$.

Canonical dimensions of quantum fields follow from the free-field Lagrangians. A scalar field $\Phi(x)$ has

$$\mathcal{L}_{\rm free} = \tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2, \tag{1}$$

so $\big[\mathcal{L}\big] = E^{+4}$, $\big[m^2\big] = E^{+2}$, and $\big[\partial_\mu\big] = E^{+1}$ imply $\big[\Phi\big] = E^{+1}$. Likewise, the EM field has

$$\mathcal{L}^{\rm EM}_{\rm free} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad\Longrightarrow\quad \big[F_{\mu\nu}\big] = E^{+2}, \tag{2}$$

and since $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the $A_\nu(x)$ field has dimension

$$\big[A_\nu\big] = \big[F_{\mu\nu}\big] \,/\, \big[\partial_\mu\big] = E^{+1}. \tag{3}$$

The massive vector fields also have $\big[A_\nu\big] = E^{+1}$ so that both terms in

$$\mathcal{L}_{\rm free} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m^2A_\nu A^\nu \tag{4}$$

have dimensions $\big[F^2\big] = \big[m^2A^2\big] = E^{+4}$.

In fact, all *bosonic* fields in 4D spacetime have canonical dimensions E^{+1} because their kinetic terms are quadratic in ∂_μ(field). On the other hand, fermionic fields line the Dirac field $\Psi(x)$ with free Lagrangian

$$\mathcal{L}_{\rm free} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \tag{5}$$

have kinetic terms with two fields but only one ∂_μ. Consequently, $\big[\mathcal{L}\big] = E^{+4}$ implies $\big[\overline{\Psi}\Psi\big] = E^{+3}$ and hence $\big[\Psi\big] = \big[\overline{\Psi}\big] = E^{+3/2}$. Similarly, all other types of fermionic fields in 4D have canonical dimension $E^{+3/2}$.

In QFTs in other spacetime dimensions $d \neq 4$, the bosonic fields such as scalars and vectors have canonical dimension

$$\big[\Phi\big] \;=\; \big[A_\nu\big] \;=\; E^{+(d-2)/2} \tag{6}$$

while the fermionic fields have canonical dimension

$$\big[\Psi\big] \;=\; E^{+(d-1)/2}. \tag{7}$$

In perturbation theory, dimensionality of coupling parameters such as λ in $\lambda\Phi^4$ theory or e in QED follows from the field's canonical dimensions. For example, in a 4D scalar theory with Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}\partial_\mu\Phi\,\partial^\mu\Phi \;-\; \tfrac{1}{2}m^2\Phi^2 \;-\sum_{n\geq3}\frac{C_n}{n!}\,\Phi^n, \tag{8}$$

the coupling C_n of the Φ^n term has dimensionality

$$\big[C_n\big] \;=\; \big[\mathcal{L}\big]\Big/\big[\Phi\big]^n \;=\; E^{4-n}. \tag{9}$$

In particular, the cubic coupling C_3 has positive energy dimension E^{+1}, the quartic coupling $\lambda = C_4$ is dimensionless, while all the higher-power couplings have negative energy dimensions E^{negative}.

Now consider a theory with a single coupling g of dimensionality $\big[g\big] = E^\Delta$. The perturbation theory in g amounts to a power series expansion

$$\mathcal{M}(\text{momenta}, g) \;=\sum_N \left(\frac{g}{\mathcal{E}^\Delta}\right)^N \times F_N(\text{momenta}) \tag{10}$$

where $\mathcal{E}$ is the overall energy scale of the process in question and all the F_N functions of momenta have the same dimensionality. The power series (10) is asymptotic rather than convergent, so it makes sense only when the expansion parameter is small,

$$\frac{g}{\mathcal{E}^\Delta} \;\ll\; 1. \tag{11}$$

For a dimensionless coupling g, this condition is simply $g \ll 1$, but for $\Delta \neq 0$, the situation is more complicated.

For couplings of positive dimensionality $\Delta > 0$, the expansion parameter (11) is always small for for high-energy processes with $\mathcal{E} \gg g^{1/\Delta}$. But for low energies $\mathcal{E} \lesssim g^{1/\Delta}$ the expansion parameter becomes large and the perturbation theory breaks down. This is a major problem for theories with $\Delta > 0$ and *massless particles.* However, if all the particles are massive, then all processes have energies $\mathcal{E} \gtrsim M_{\text{lightest}}$, and this makes couplings with $\Delta > 0$ OK as long as

$$g \ll M_{\text{lightest}}^{\Delta} . \tag{12}$$

Couplings of negative dimensionality $\Delta < 0$ have an opposite problem: The expansion parameter (11) is small at low energies but becomes large at high energies $\mathcal{E} \gtrsim g^{-1/\Delta}$. Beyond the maximal energy

$$E^{\max} \sim g^{-1/\Delta}, \tag{13}$$

the perturbation theory breaks down and we may no longer compute the S–matrix elements $\mathcal{M}$ using any finite number of Feynman diagrams.

Worse, in Feynman diagrams with loops one must worry not only about energies of the incoming and outgoing particles but also about momenta q^μ of the internal lines. Basically, an L–loop diagram contributing to N^{th} term in the expansion (10) produces something like

$$g^N \times \int d^{4L}q \mathcal{F}_N(q,p,k,m) \qquad \text{where} \quad \big[\mathcal{F}_N\big] = E^{-N\Delta-4L+\text{C}}, \quad C = \text{const.} \tag{14}$$

For very large loop momenta $q \gg p,k,m$, dimensionality implies $\mathcal{F}_N \propto q^{-N\Delta-4L+\text{C}}$, so for $-N\Delta + C \geq 0$, the integral (14) diverges as $q \to \infty$. Moreover, for $\Delta < 0$ higher orders of perturbation theory have worse divergences of increasing degrees $-N\Delta + C \geq 0$. Therefore, *field theories with $\Delta < 0$ couplings do not work as complete theories.*

However, theories with $\Delta < 0$ may be used as approximate *effective theories* (without the divergent loop graphs) for low-energy processes, $\mathcal{E} \lesssim \Lambda$ for some $\Lambda < g^{-1/\Delta}$. For example, Fermi theory of weak interactions

$$\mathcal{L}_{\text{int}} = 2\sqrt{2}G_f \times J_\mu^+ J^{\mu-} \qquad \text{where} \quad J_\mu^{\pm} = \sum_{\substack{\text{appropriate}\\ \text{fermions}}} \overline{\Psi}\frac{1-\gamma^5}{2}\gamma_\mu\Psi \tag{15}$$

has coupling G_F of dimension $\big[G_G\big] = E^{-2}$; its value is $G_F \approx 1.17 \cdot 10^{-5}\,\text{GeV}^{-2}$. This is a good effective theory for low-energy weak interactions, but it cannot be used for energies

$\mathcal{E} \gtrsim 1/\sqrt{G_F} \sim 300$ GeV, not even theoretically. In real life, Fermi theory works well for $\mathcal{E} \ll M_W \sim 80$ GeV, but for higher energies one should use the complete $SU(2) \times U(1)$ electroweak theory including $W^\pm$ and Z^0 particles, *etc.*

Similar to the Fermi theory, most effective theories with $\Delta < 0$ couplings are low-energy limits of more complicated theories with extra heavy particles of masses $M \lesssim g^{-1/\Delta}$ but no $\Delta < 0$ couplings.

In QFTs which are valid for all energies, all coupling must have zero or positive energy dimensions. In 4D, a coupling involving b bosonic fields (scalar or vector), f fermionic fields, and δ derivatives ∂_μ has dimensionality

$$\Delta \;=\; 4 \;-\; b \;-\; \tfrac{3}{2} f \;-\; \delta. \tag{16}$$

Thus, only the boson^3 couplings have $\Delta > 0$ while the $\Delta = 0$ couplings comprise boson^4, boson $\times$ fermion^2, and boson^2 $\times$ ∂boson. All other coupling types have $\Delta > 0$ and are not allowed (except in effective theories).

Here is the complete list of the allowed couplings in 4D.

1. Scalar couplings

$$-\frac{\mu}{3!}\,\Phi^3 \quad \text{and} \quad -\frac{\lambda}{4!}\,\Phi^4. \tag{17}$$

   Note: the higher powers Φ^5, Φ^6, *etc.*, are not allowed because the couplings would have $\Delta < 0$.

2. Gauge couplings of vectors to charged scalars

$$-iqA^\mu\left(\Phi^*\partial_\mu\Phi \;-\; \Phi\partial_\mu\Phi^*\right) \;+\; q^2\Phi^*\Phi A_\mu A^\mu \;\subset\; D_\mu\Phi^*\, D^\mu\Phi. \tag{18}$$

3. Non-abelian gauge couplings between the vector fields

$$-gf^{abc}(\partial_\mu A^a_\nu)A^{\mu b}A^{\nu c} \;-\; \frac{g^2}{4}\,f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} \;\subset\; -\frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu a}. \tag{19}$$

4. Gauge couplings of vectors to charged fermions,

$$-qA^\mu \times \overline{\Psi}\gamma_\mu\Psi \;\subset\; \overline{\Psi}(i\gamma_\mu D^\mu)\Psi. \tag{20}$$

If the fermions are massless and chiral, we may also have

$$-qA_\mu \times \overline{\Psi}\frac{1 \pm \gamma^5}{2}\,\gamma_\mu \Psi, \tag{21}$$

or in Weyl fermion language

$$-qA_\mu \times \chi^\dagger \bar{\sigma}_\mu \chi.$$

5. Yukawa couplings of scalars to fermions,

$$-g\Phi \times \overline{\Psi}\Psi \quad \text{or} \quad -ig\Phi \times \overline{\Psi}\gamma^5\Psi. \tag{22}$$

If parity is conserved, in the first term Φ should be a true scalar, and in the second term a pseudo-scalar.