Interaction Picture and Dyson Series

Many quantum systems have Hamiltonians of the form $\hat{H} = \hat{H}_0 + \hat{V}$ where $\hat{H}_0$ is a *free Hamiltonian* with known spectrum — which is used to classify the states of the full theory — while $\hat{V}$ is a *perturbation* which causes transitions between the eigenstates of the $\hat{H}_0$. For example, in scattering theory

$$\hat{H}_0 \;=\; \frac{\hat{\mathbf{P}}^2_{\rm red}}{2M_{\rm red}}\,, \qquad \hat{V} \;=\; \text{potential } V(\hat{\mathbf{x}}_{\rm rel}). \tag{1}$$

Similarly, for a self-interacting quantum scalar field we have

$$\begin{aligned}\hat{H}_0 \;&=\; \int\! d^3\mathbf{x} \left(\tfrac12\hat{\Pi}^2(\mathbf{x}) \,+\, \tfrac12\left(\nabla\hat{\Phi}(\mathbf{x})\right)^2 \,+\, \frac{m^2}{2}\,\hat{\Phi}^2\right)\\ &=\; \int \frac{d^3\mathbf{p}}{16\pi^3 E_{\mathbf{p}}}\, E_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} \;+\; \text{const},\\ \hat{V} \;&=\; \int\! d^3\mathbf{x}\, \frac{\lambda}{24}\,\hat{\Phi}^4(\mathbf{x}).\end{aligned} \tag{2}$$

To study the transitions (scattering, making new particles, decays, *etc.*) caused by $\hat{V}$ we want to use a fixed basis of $\hat{H}_0$ eigenstates, but we want to keep the transitions separate from wave-packet spreading and other effects due to Schrödinger phases e^{-iEt} of the $\hat{H}_0$ itself. The picture of QM which separates these effects is the *interaction picture.*

In the Schrödinger picture, the operators are time-independent while the quantum states evolve with time as $|\psi\rangle_S(t) = e^{-i\hat{H}t}\,|\psi\rangle(0)$. In the Heisenberg picture it's the other way around — the quantum states are time independent while the operators evolve with time — and the two pictures are related by a time-dependent unitary operator $e^{i\hat{H}t}$,

$$|\Psi\rangle_H(t) \;=\; e^{+i\hat{H}t}\,|\Psi\rangle_S(t) \;\equiv\; |\Psi\rangle_S(0)\ \forall t, \qquad \hat{A}_H \;=\; e^{+i\hat{H}t}\hat{A}_S e^{-i\hat{H}t}. \tag{3}$$

The interaction picture has a similar relation to the Schrödinger's, but using the $e^{i\hat{H}_0 t}$ instead of the $e^{i\hat{H}t}$,

$$\begin{aligned}\hat{A}_I \;&=\; e^{+i\hat{H}_0 t}\hat{A}_S e^{-\hat{H}_0 t},\\ |\psi\rangle_I(t) \;&=\; e^{+i\hat{H}_0 t}\,|\psi\rangle_S(t) \;=\; e^{+i\hat{H}_0 t}e^{-i\hat{H}t}\,|\psi\rangle_H \;\neq\; \text{const}.\end{aligned} \tag{4}$$

In the interaction picture, quantum fields depend on time as if they were free fields, for example

$$\hat{\Phi}_I(\mathbf{x},t) \;=\; \int \frac{d^3\mathbf{p}}{16\pi^3 E_{\mathbf{p}}} \left(e^{-ipx}\hat{a}_p \,+\, e^{+ipx}\hat{a}_p^\dagger \right)^{p^0=+E_{\mathbf{p}}}, \tag{5}$$

regardless of the interactions. This is different form the Heisenberg picture where non-free fields depend on time in a much more complicated way.

In the interaction picture, time-dependence of the quantum states is governed by the perturbation $\hat{V}$ according to Schrödinger-like equation

$$i\frac{d}{dt}\left|\psi\right\rangle_I(t) \;=\; \hat{V}_I(t)\,\left|\Psi\right\rangle_I(t). \tag{6}$$

The problem with this equation is that the $\hat{V}_I$ operator here is itself in the interaction picture, so it depends on time as $\hat{V}_I(t) = e^{+i\hat{H}_0 t}\hat{V}_S e^{-\hat{H}_0 t}$. Consequently, the *evolution operator* for the interaction picture

$$\hat{U}_I(t,t_0): \quad \left|\psi\right\rangle_I(t) \;=\; \hat{U}_I(t,t_0)\left|\psi\right\rangle_I(t_0) \tag{7}$$

is much more complicated than simply $e^{-i\hat{V}(t-t_0)}$.

The evolution operator satisfies

$$i\frac{\partial}{\partial t}\hat{U}_I(t,t_0) \;=\; \hat{V}_I(t)\,\hat{U}_I(t,t_0), \qquad \hat{U}_I(t=t_0) \;=\; 1, \tag{8}$$

and the formal solution to this equation is the **Dyson series**

$$\begin{aligned}\hat{U}_I(t,t_0) \;&=\; 1 \,-\, i\int_{t_0}^{t} dt_1\,\hat{V}_I(t_1) \,-\, \int_{t_0}^{t} dt_2\,\hat{V}_I(t_2)\int_{t_0}^{t_2} dt_1\,\hat{V}_I(t_1)\\ &\qquad +\, i\int_{t_0}^{t} dt_3\,\hat{V}_I(t_3)\int_{t_0}^{t_3} dt_2\,\hat{V}_I(t_2)\int_{t_0}^{t_2} dt_1\,\hat{V}_I(t_1) \,+\, \cdots\\ &=\; 1 \,+\, \sum_{n=1}^{\infty}(-i)^n \int\!\!\cdots\!\!\int_{t_0<t_1<\cdots<t_n<t} dt_n\cdots dt_1\,\hat{V}_I(t_n)\cdots\hat{V}_I(t_1).\end{aligned} \tag{9}$$

Note time ordering of operators $\hat{V}_I(t_n)\cdots\hat{V}_I(t_1)$ in each term.

To see that the Dyson series satisfies eqs. (8), we note that in each term, the only thing which depends on t is the upper limit of the leftmost dt_n integral. Thus, taking $\partial/\partial t$ of the term amounts to skipping that integral and letting $t_n = t$,

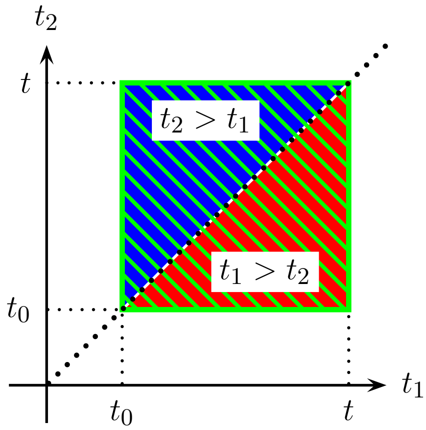
$$
\begin{aligned}
i\frac{\partial}{\partial t}\left((-i)^n \int\limits_{t_0}^{t} dt_n\, \hat{V}_I(t_n) \int\limits_{t_0}^{t_n} dt_{n-1}\, \hat{V}_I(t_{n-1}) \,\cdots \int\limits_{t_0}^{t_2} dt_1\, \hat{V}_I(t_1)\right) \; &= \\
= \; \hat{V}_I(t_n = t) \times \left((-i)^{n-1} \int\limits_{t_0}^{t_n=t} dt_{n-1}\, \hat{V}_I(t_{n-1}) \,\cdots \int\limits_{t_0}^{t_2} dt_1\, \hat{V}_I(t_1)\right).&
\end{aligned}
\tag{10}
$$

In other words, $i\partial/\partial t$ of the n^{th} term is $\hat{V}_I(t) \times$ the $(n-1)^{\text{st}}$ term, and that's how eq. (8) is satisfied. And the initial condition $\hat{U}_I(t = t_0) = 1$ is also satisfied (this is obvious).

Thanks to the time ordering of the $\hat{V}_I(t)$ in each term of the Dyson series — the earliest operator being rightmost so it acts first, the second earliest being second from the right, *etc.*, until the latest operator stands to the left of everything so it acts last — we may rewrite the integrals in a more compact form using the time-orderer $\mathbf{T}$. Earlier in class, I have defined $\mathbf{T}$ of an operator product, but now I would like to extend this by linearity to any sum of operators products. Similarly, we may time-order integrals of operator products and hence products of integrals such as

$$
\begin{aligned}
\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_I(t')\right)^2 \;&\stackrel{\text{def}}{=}\; \mathbf{T} \iint\limits_{t_0}^{t} dt_1\, dt_2\, \hat{V}_I(t_1)\hat{V}_I(t_2) \;\stackrel{\text{def}}{=}\; \iint\limits_{t_0}^{t} dt_1\, dt_2\, \mathbf{T}\hat{V}_I(t_1)\hat{V}_I(t_2) \\
&= \iint\limits_{t_0<t_1<t_2<t} dt_1\, dt_2\, \hat{V}_I(t_2)\hat{V}_I(t_1) \;+\; \iint\limits_{t_0<t_2<t_1<t} dt_1\, dt_2\, \hat{V}_I(t_1)\hat{V}_I(t_2)
\end{aligned}
\tag{11}
$$

where the domain of each $dt_1\, dt_2$ integral is color-coded on the diagram below:



(12)

Integral over the blue triangle $t_0 < t_1 < t_2 < t$ is what appears in the Dyson series. But there is $t_1 \leftrightarrow t_2$ symmetry between the blue and red triangles, and the corresponding integrals on the bottom line of eq. (11) are equal to each other. Hence the triangular integral in the Dyson series may be written in a more compact form as

$$\iint\limits_{t_0<t_1<t_2<t} dt_1\, dt_2\, \hat{V}_I(t_2)\hat{V}_I(t_1) \;=\; \frac{1}{2}\,\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_1(t')\right)^2 . \tag{13}$$

Similar procedure applies to the higher-order terms in the Dyson series. The n^{th} order term is an integral over a simplex $t_0 < t_1 < t_2 < \cdots < t_n < t$ in the $(t_1, \ldots, t_n)$ space. A hypercube $t_0 < t_1, \ldots, t_n < t$ contains $n!$ such simplexes, and after time-ordering the $\hat{V}$ operators, integrals over all simplexes become equal by permutation symmetry. Thus,

$$\begin{aligned}
\int\!\!\cdots\!\!\int\limits_{t_0<t_1<\cdots<t_n<t} dt_n \cdots dt_1\, \hat{V}_I(t_n)\cdots\hat{V}_I(t_1) \;&=\; \frac{1}{n!}\int\limits_{t_0}^{t}\!\!\cdots\!\!\int dt_n \cdots dt_1\, \mathbf{T}\hat{V}_I(t_n)\cdots\hat{V}_I(t_1) \\
&=\; \frac{1}{n!}\,\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_1(t')\right)^n .
\end{aligned} \tag{14}$$

Altogether, the Dyson series becomes a *time-ordered exponential*

$$\hat{U}_I(t,t_0) \;=\; 1 \;+\; \sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\mathbf{T}\left(\int\limits_{t_0}^{t} dt'\, \hat{V}_I(t')\right)^n \;\equiv\; \mathbf{T}\text{-exp}\left(-i\int\limits_{t_0}^{t} dt'\, \hat{V}_I(t')\right). \tag{15}$$

Of particular interest is the evolution operator from distant past to distant future,

$$\hat{S} \stackrel{\text{def}}{=} \hat{U}_I(+\infty,-\infty) \;=\; \mathbf{T}\text{-exp}\left(-i\int\limits_{-\infty}^{+\infty} dt'\,\hat{V}_I(t')\right). \tag{16}$$

This operator is properly called 'the scattering operator' or 'the S–operator', but everybody calls it 'the S–matrix'. In the scalar field theory where

$$\hat{V}_I(t) \;=\; \frac{\lambda}{24}\int d^3\mathbf{x}\,\hat{\Phi}_I^4(\mathbf{x},t), \tag{17}$$

the S–matrix has a Lorentz-invariant form

$$\hat{S} \;=\; \mathbf{T}\text{-exp}\left(\frac{-i\lambda}{24}\int\limits_{\substack{\text{whole}\\ \text{spacetime}}} d^4x\,\hat{\Phi}_I^4(x)\right). \tag{18}$$

Note that $\hat{\Phi}_I(x)$ here is the *free* scalar field as in eq. (5). Similar Lorentz-invariant expressions exist for other quantum field theories. For example, in QED

$$\hat{S} \;=\; \mathbf{T}\text{-exp}\left(+ie\int\limits_{\substack{\text{whole}\\ \text{spacetime}}} d^4x\,\hat{A}_I^\mu(x)\hat{\overline{\Psi}}_I(x)\gamma_\mu\hat{\Psi}_I(x)\right) \tag{19}$$

where both the EM field $\hat{A}_I^\mu(x)$ and the electron field $\Psi_I(x)$ are in the interaction picture so they evolve with time as free fields.

Alas, eqs. (18), (19), and similar formulae for other quantum field theories do not help us to evaluate the S-matrix's *elements* $\langle\text{out}|\,\hat{S}\,|\text{in}\rangle$ between *physical* incoming and outgoing 2–particle (or n-particle) states. In the potential scattering theory the asymptotic states are simply eigenstates of the free Hamiltonian, but this does not work in QFT. Unfortunately, asymptotic n–particle states $|p_1,\ldots,p_n\rangle$ of the interacting field theory are quite different from the free theory's n–particle states $\hat{a}^\dagger_{\mathbf{p}_n}\cdots\hat{a}^\dagger_{\mathbf{p}_1}|0\rangle$. Even the physical vacuum $|\Omega\rangle$ is different from the free theory's vacuum $|0\rangle$.

But there is another way of calculating the physical S–matrix elements $\langle\text{out}|\,\hat{S}\,|\text{in}\rangle$; instead of eqs. (18), (19), *etc.*, it uses correlation functions of the fully-interacting quantum fields.

Correlation Functions

The n-point correlation function of the scalar field theory is defined as

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle\,. \tag{20}$$

Note that all the fields $\hat{\Phi}_H(x)$ here are in the Heisenberg picture so their time dependence involves the complete Hamiltonian $\hat{H}$ rather than just the $\hat{H}_0$. Likewise, $|\Omega\rangle$ is the ground state of $\hat{H}$, *i.e.* the true physical vacuum of the interacting theory rather than the free theory's vacuum $|0\rangle$.

Other quantum field theories with fields $\hat{\phi}^\alpha$ (which could be scalar, vector, tensor, spinor, whatever) have similar correlation functions

$$\mathcal{F}^{\alpha_1,\ldots,\alpha_n}(x_1,\ldots,x_n) \;=\; \langle\Omega|\,\mathbf{T}\hat{\phi}_H^{\alpha_1}(x_1)\cdots\hat{\phi}_H^{\alpha_n}(x_n)\,|\Omega\rangle\,. \tag{21}$$

Again, all the $\phi_H^{\alpha_i}(x_i)$ are in the Heisenberg picture, so they are interacting rather than free fields. But for now, let's focus on the theory of a single scalar fields and its correlation functions (20).

In perturbation theory, the correlation functions $\mathcal{F}_n$ of the interacting theory are related to the free correlation functions

$$\langle 0|\,\mathbf{T}\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\cdots\text{more }\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots|0\rangle\,. \tag{22}$$

involving additional fields $\hat{\Phi}_I(z_1)\hat{\Phi}_I(z_2)\cdots$. Note that in eq. (22) the fields are in the interaction rather than Heisenberg picture, so they evolve with time as free fields. Likewise, $|0\rangle$ is the free theory's vacuum, *i.e.* the ground state of the free Hamiltonian $\hat{H}_0$ rather than the full Hamiltonian $\hat{H}$.

To work out the relation between (20) and (22), we start by formally relating quantum fields in the Heisenberg and the interaction pictures,

$$\hat{\Phi}_H(\mathbf{x},t) \;=\; e^{+i\hat{H}t}\hat{\Phi}_S(\mathbf{x})e^{-i\hat{H}t} \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0t}\hat{\Phi}_I(\mathbf{x},t)e^{+i\hat{H}_0t}e^{-i\hat{H}t}. \tag{23}$$

We may re-state this relation in terms of evolution operators using a formal expression for the later,

$$\hat{U}_I(t,t_0) \;=\; e^{+i\hat{H}_0t}e^{-i\hat{H}(t-t_0)}e^{-i\hat{H}_0t_0}. \tag{24}$$

Note that this formula applies for both forward and backward evolution, *i.e.* regardless of whether $t > t_0$ or $t < t_0$. In particular,

$$\hat{U}_I(t,0) \;=\; e^{+i\hat{H}_0t}e^{-i\hat{H}t} \qquad \text{and} \qquad \hat{U}_I(0,t) \;=\; e^{+i\hat{H}t}e^{-i\hat{H}_0t}, \tag{25}$$

which allows us to re-state eq. (23) as

$$\hat{\Phi}_H(x) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,0). \tag{26}$$

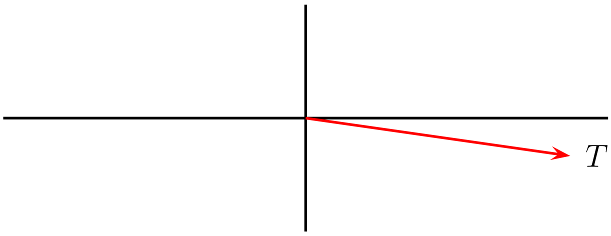
Consequently,

$$\hat{\Phi}_H(x)\hat{\Phi}_H(y) \;=\; \hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0) \tag{27}$$

because $\hat{U}_I(x^0,0)\hat{U}_I(0,y^0) = \hat{U}_I(x^0,y^0)$, and likewise for n fields

$$\begin{aligned}\hat{\Phi}_H(x_1)\hat{\Phi}_H(x_2)\cdots\hat{\Phi}_H(x_n) \;=&\\ =\; \hat{U}_I(0,x_1^0)\hat{\Phi}_I(x_1)\hat{U}_I(x_1^0,x_2^0)\hat{\Phi}_I(x_2)&\cdots\hat{U}_I(x_{n-1}^0,x_n^0)\hat{\Phi}_I(x_n)\hat{U}_I(x_n^0,0).\end{aligned} \tag{28}$$

Now we need to relate the free vacuum $|0\rangle$ and the true physical vacuum $|\Omega\rangle$. Consider the state $\hat{U}_I(0,-T)\,|0\rangle$ for a complex T, and take the limit of $T \to (+1-i\epsilon)\times\infty$. That is, $\mathrm{Re}\, T \to +\infty$, $\mathrm{Im}\, T \to -\infty$, but the imaginary part grows slower than the real part. Pictorially, in the complex T plane,



(29)

we go infinitely far to the right at infinitesimally small angle below the real axis.

Without loss of generality we assume the free theory has zero vacuum energy, thus $\hat{H}_0 |0\rangle = 0$ and hence

$$\hat{U}_I(0, -T) |0\rangle \;=\; e^{-i\hat{H}T} e^{+i\hat{H}_0 T} |0\rangle \;=\; e^{-i\hat{H}T} |0\rangle . \tag{30}$$

From the interacting theory's point of view, $|0\rangle$ is a superposition of eigenstates $|Q\rangle$ of the full Hamiltonian $\hat{H}$,

$$|0\rangle \;=\; \sum_Q |Q\rangle \times \langle Q|0\rangle \quad\Longrightarrow\quad e^{-i\hat{H}T} |0\rangle \;=\; \sum_Q |Q\rangle \times e^{-iTE_Q} \langle Q|0\rangle \tag{31}$$

In the $T \to (+1 - i\epsilon) \times \infty$ limit, the second sum here is dominated by the term with the lowest E_Q, so we look for the lowest energy eigenstate $|Q_0\rangle$ with the same quantum numbers as $|0\rangle$ (otherwise, we would have zero overlap $\langle Q_0|0\rangle$). Obviously, such $|Q_0\rangle$ is the physical vacuum $|\Omega\rangle$, so

$$\hat{U}_I(0, -T) |0\rangle \;\xrightarrow[T \to (+1 - i\epsilon)\infty]{}\; |\Omega\rangle \times e^{-iTE_\Omega} \langle \Omega|0\rangle \tag{32}$$

and therefore

$$|\Omega\rangle \;=\; \lim_{T \to (+1 - i\epsilon)\infty} \hat{U}_I(0, -T) |0\rangle \times \frac{e^{+iTE_\Omega}}{\langle \Omega|0\rangle} . \tag{33}$$

Likewise,

$$\langle \Omega| \;=\; \lim_{T \to (+1 - i\epsilon)\infty} \frac{e^{+iTE_\Omega}}{\langle 0|\Omega\rangle} \times \langle 0| \hat{U}_I(+T, 0) . \tag{34}$$

Combining eqs. (27), (33), and (34), we may now express a two-point function as

$$\langle\Omega|\,\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;=\; \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\text{Big\_Product}\,|0\rangle \tag{35}$$

where

$$C(T) \;=\; \frac{e^{2iTE_\Omega}}{|\,\langle 0|\Omega\rangle\,|^2} \tag{36}$$

is a just a coefficient, and

$$\begin{aligned}\text{Big\_Product} \;&=\; \hat{U}_I(+T,0)\hat{U}_I(0,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,0)\hat{U}_I(0,-T)\\ &=\; \hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T).\end{aligned} \tag{37}$$

For $x^0 > y^0$ the last line here is in proper time order, so if we re-order the operators, the time-orderer $\mathbf{T}$ would put them back where they belong. Thus, using $\mathbf{T}$ to keep track of the operator order, we have

$$\begin{aligned}\text{Big\_Product} \;&=\; \mathbf{T}\Big(\hat{U}_I(+T,x^0)\hat{\Phi}_I(x)\hat{U}_I(x^0,y^0)\hat{\Phi}_I(y)\hat{U}_I(y^0,-T)\Big)\\ &=\; \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,x^0)\hat{U}_I(x^0,y^0)\hat{U}_I(y^0,-T)\Big)\\ &=\; \mathbf{T}\Big(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\hat{U}_I(+T,-T)\Big)\\ &=\; \mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int_{-T}^{+T}\!dt\int\!d^3\mathbf{z}\,\hat{\Phi}_I^4(t,\mathbf{z})\right)\right).\end{aligned} \tag{38}$$

Therefore

$$\langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x)\hat{\Phi}_H(y)\,|\Omega\rangle \;= \tag{39}$$
$$=\; \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x)\hat{\Phi}_I(y)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,,$$

where the spacetime integral has ranges

$$\int\!d^4z \;\equiv\; \int_{-T}^{+T}\!dz^0\int_{\substack{\text{whole}\\\text{space}}} d^3\mathbf{z}\,. \tag{40}$$

Similarly, the n-point correlation functions are given by

$$\begin{aligned}\mathcal{F}_n(x_1,\ldots,x_n) \;&=\\ &= \langle\Omega|\,\mathbf{T}\hat{\Phi}_H(x_1)\cdots\hat{\Phi}_H(x_n)\,|\Omega\rangle\\ &= \lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle\,.\end{aligned} \tag{41}$$

Note that the coefficient $C(T)$ — *cf.* eq. (36) — is the same for all correlation functions. In particular, for $n=0$ the $\mathcal{F}_0=\langle\Omega|\Omega\rangle=1$, but it's also given by eq. (41), hence

$$\lim_{T\to(+1-i\epsilon)\infty} C(T)\times\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle \;=\; 1. \tag{42}$$

This allows us to eliminate the $C(T)$ factors from eqs. (41) by taking *ratios* of the free-theory correlation functions,

$$\mathcal{F}_n(x_1,\ldots,x_n) \;=\; \lim_{T}\frac{\langle 0|\,\mathbf{T}\left(\hat{\Phi}_I(x_1)\cdots\hat{\Phi}_I(x_n)\times\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle}{\langle 0|\,\mathbf{T}\left(\exp\left(\frac{-i\lambda}{24}\int\!d^4z\,\hat{\Phi}_I^4(z)\right)\right)|0\rangle}\,. \tag{43}$$

The limit here is $T\to(+1-i\epsilon)\times\infty$, and the T dependence under the limit is implicit in the ranges of the spacetime integrals, *cf.* eq. (40).