

1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - A^\mu J_\mu \quad (1)$$

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

- (a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.
- (b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu = 0 \quad \text{and} \quad (\partial^2 + m^2)A^\mu = J^\mu. \quad (2)$$

2. In string theory, one often has to work with tensor fields in spacetimes of high dimensions $D > 4$, such as $D = 9 + 1$ or even $D = 25 + 1$. In this exercise, we consider a free antisymmetric tensor field $B_{\mu\nu}(x) \equiv -B_{\nu\mu}(x)$, where μ and ν are D -dimensional Lorentz indices running from 0 to $D - 1$.

To be precise, $B_{\mu\nu}(x)$ is the *tensor potential*, analogous to the electromagnetic vector potential $A_\mu(x)$. The analog of the EM tension fields $F_{\mu\nu}(x)$ is the 3-index tension tensor

$$H_{\lambda\mu\nu}(x) = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}. \quad (3)$$

- (a) Show that this tensor is totally antisymmetric in all 3 indices.

(b) Show that regardless of the Lagrangian, the H fields satisfy Jacobi identities

$$\frac{1}{6}\partial_{[\kappa}H_{\lambda\mu\nu]} \equiv \partial_{\kappa}H_{\lambda\mu\nu} - \partial_{\lambda}H_{\mu\nu\kappa} + \partial_{\mu}H_{\nu\kappa\lambda} - \partial_{\nu}H_{\kappa\lambda\mu} = 0. \quad (4)$$

(c) The Lagrangian for the $B_{\mu\nu}(x)$ fields is

$$\mathcal{L}(B, \partial B) = \frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \quad (5)$$

where $H_{\lambda\mu\nu}$ are as in eq. (3). Treating the $B_{\mu\nu}(x)$ as $\frac{1}{2}D(D-1)$ independent fields, derive their equations of motion.

Similar to the EM fields, the $B_{\mu\nu}$ fields are subject to *gauge transforms*

$$B'_{\mu\nu}(x) = B_{\mu\nu}(x) + \partial_{\mu}\Lambda_{\nu}(x) - \partial_{\nu}\Lambda_{\mu}(x) \quad (6)$$

where $\Lambda_{\mu}(x)$ is an arbitrary vector field.

(d) Show that the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (5) — are invariant under such gauge transforms.

In spacetimes of sufficiently high dimensions D , one may have similar tensor fields with more indices. Generally, the potentials form a p -index totally antisymmetric tensor $C_{\mu_1\mu_2\cdots\mu_p}(x)$, the tensions form a $p+1$ index tensor

$$G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) = \frac{1}{p!}\partial_{[\mu_1}C_{\mu_2\cdots\mu_p\mu_{p+1}]}(x), \quad (7)$$

also totally antisymmetric in all its indices, and the Lagrangian is

$$\mathcal{L}(C, \partial C) = \frac{(-1)^p}{2(p+1)!}G_{\mu_1\mu_2\cdots\mu_{p+1}}G^{\mu_1\mu_2\cdots\mu_{p+1}}. \quad (8)$$

(e) Derive the Jacobi identities and the equations of motion for the G fields.

(f) Show that the tension fields $G_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ — and hence the Lagrangian (8) — are invariant under gauge transforms of the potentials $C_{\mu_1\mu_2\cdots\mu_p}(x)$ which act as

$$C'_{\mu_1\mu_2\cdots\mu_p}(x) = C_{\mu_1\mu_2\cdots\mu_p}(x) + \frac{1}{(p-1)!} \partial_{[\mu_1} \Lambda_{\mu_2\cdots\mu_p]}(x) \quad (9)$$

where $\Lambda_{\mu_2\cdots\mu_p}(x)$ is an arbitrary $(p-1)$ -index tensor field (totally antisymmetric).

3. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T_{\text{Noether}}^{\mu\nu} = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - g^{\mu\nu} \mathcal{L}. \quad (10)$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}, \quad (11)$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3-index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi, \partial\phi)$ as a function of the fields and their derivatives,

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \partial_\mu T_{\text{Noether}}^{\mu\nu} = (\text{hopefully}) = 0 \\ P_{\text{net}}^\mu &\equiv \int d^3\mathbf{x} T^{0\mu} = \int d^3\mathbf{x} T_{\text{Noether}}^{0\mu}. \end{aligned} \quad (12)$$

For the scalar fields, real or complex, $T_{\text{Noether}}^{\mu\nu}$ is properly symmetric and one simply has $T^{\mu\nu} = T_{\text{Noether}}^{\mu\nu}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu, \partial_\nu A_\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (13)$$

where A_μ is a real vector field and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

- (b) Write down $T_{\text{Noether}}^{\mu\nu}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.
- (c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T_{\text{EM}}^{\mu\nu} = -F^{\mu\lambda}F^\nu{}_\lambda + \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}. \quad (14)$$

Show that this expression indeed has form (11) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

- (d) Write down the components of the stress-energy tensor (14) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Now consider the electromagnetic fields coupled to the electric current J^μ of some charged “matter” fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate P_{EM}^μ and P_{mat}^μ . Consequently, we should have

$$\partial_\mu T_{\text{net}}^{\mu\nu} = 0 \quad \text{for} \quad T_{\text{net}}^{\mu\nu} = T_{\text{EM}}^{\mu\nu} + T_{\text{mat}}^{\mu\nu} \quad (15)$$

but generally $\partial_\mu T_{\text{EM}}^{\mu\nu} \neq 0$ and $\partial_\mu T_{\text{mat}}^{\mu\nu} \neq 0$.

- (e) Use Maxwell’s equations to show that

$$\partial_\mu T_{\text{EM}}^{\mu\nu} = -F^{\nu\lambda}J_\lambda \quad (16)$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T_{\text{mat}}^{\mu\nu} = +F^{\nu\lambda}J_\lambda. \quad (17)$$