1. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{1}$$

where $c = \hbar = 1$, $F_{\mu\nu} \stackrel{\rm def}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$, and the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Note that because of the mass term, the Lagrangian (1) is *not* gauge invariant.

(a) Derive the Euler–Lagrange field equations for the massive vector field $A^\mu(x)$.

(b) Show that this field equation *does not require* current conservation; however, if the current happens to satisfy $\partial_\mu J^\mu = 0$, then the field $A^\mu(x)$ satisfies

$$\partial_\mu A^\mu \;=\; 0 \qquad \text{and} \qquad (\partial^2 + m^2)A^\mu = J^\mu. \tag{2}$$

2. In string theory, one often has to work with tensor fields in spacetimes of high dimensions $D > 4$, such as $D = 9+1$ or even $D = 25+1$. In this exercise, we consider a free antisymmetric tensor field $B_{\mu\nu}(x) \equiv -B_{\nu\mu}(x)$, where μ and ν are D-dimensional Lorentz indices running from 0 to $D-1$.

To be precise, $B_{\mu\nu}(x)$ is the *tensor potential*, analogous to the electromagnetic vector potential $A_\mu(x)$. The analog of the EM tension fields $F_{\mu\nu}(x)$ is is the 3–index tension tensor

$$H_{\lambda\mu\nu}(x) \;=\; \partial_\lambda B_{\mu\nu} \;+\; \partial_\mu B_{\nu\lambda} \;+\; \partial_\nu B_{\lambda\mu}\,. \tag{3}$$

(a) Show that this tensor is totally antisymmetric in all 3 indices.

(b) Show that regardless of the Lagrangian, the H fields satisfy Jacobi identities

$$\tfrac{1}{6}\partial_{[\kappa}H_{\lambda\mu\nu]} \;\equiv\; \partial_\kappa H_{\lambda\mu\nu} \;-\; \partial_\lambda H_{\mu\nu\kappa} \;+\; \partial_\mu H_{\nu\kappa\lambda} \;-\; \partial_\nu H_{\kappa\lambda\mu} \;=\; 0. \tag{4}$$

(c) The Lagrangian for the $B_{\mu\nu}(x)$ fields is

$$\mathcal{L}(B,\partial B) \;=\; \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \tag{5}$$

where $H_{\lambda\mu\nu}$ are as in eq. (3). Treating the $B_{\mu\nu}(x)$ as $\frac{1}{2}D(D-1)$ independent fields, derive their equations of motion.

Similar to the EM fields, the $B_{\mu\nu}$ fields are subject to *gauge transforms*

$$B'_{\mu\nu}(x) \;=\; B_{\mu\nu}(x) \;+\; \partial_\mu\Lambda_\nu(x) \;-\; \partial_\nu\Lambda_\mu(x) \tag{6}$$

where $\Lambda_\mu(x)$ is an arbitrary vector field.

(d) Show that the tension fields $H_{\lambda\mu\nu}(x)$ — and hence the Lagrangian (5) — are invariant under such gauge transforms.

In spacetimes of sufficiently high dimensions D, one may have similar tensor fields with more indices. Generally, the potentials form a p-index totally antisymmetric tensor $C_{\mu_1\mu_2\cdots\mu_p}(x)$, the tensions form a $p+1$ index tensor

$$G_{\mu_1\mu_2\cdots\mu_{p+1}}(x) \;=\; \frac{1}{p!}\,\partial_{[\mu_1}C_{\mu_2\cdots\mu_p\mu_{p+1}]}(x), \tag{7}$$

also totally antisymmetric in all its indices, and the Lagrangian is

$$\mathcal{L}(C,\partial C) \;=\; \frac{(-1)^p}{2(p+1)!}\,G_{\mu_1\mu_2\cdots\mu_{p+1}}G^{\mu_1\mu_2\cdots\mu_{p+1}}. \tag{8}$$

(e) Derive the Jacobi identities and the equations of motion for the G fields.

(f) Show that the tension fields $G_{\mu_1\mu_2\cdots\mu_{p+1}}(x)$ — and hence the Lagrangian (8) — are invariant under gauge transforms of the potentials $C_{\mu_1\mu_2\cdots\mu_p}(x)$ which act as

$$C'_{\mu_1\mu_2\cdots\mu_p}(x) \;=\; C_{\mu_1\mu_2\cdots\mu_p}(x) \;+\; \frac{1}{(p-1)!}\,\partial_{[\mu_1}\Lambda_{\mu_2\cdots\mu_p]}(x) \tag{9}$$

where $\Lambda_{\mu_2\cdots\mu_p}(x)$ is an arbitrary $(p-1)$-index tensor field (totally antisymmetric).

3. According to the Noether theorem, a translationally invariant system of classical fields $\phi_a(x)$ has a conserved stress-energy tensor

$$T^{\mu\nu}_{\text{Noether}} \;=\sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi^a \;-\; g^{\mu\nu}\,\mathcal{L}. \tag{10}$$

Actually, to assure the symmetry of the stress-energy tensor, $T^{\mu\nu} = T^{\nu\mu}$ (which is necessary for the angular momentum conservation), one sometimes has to add a total divergence,

$$T^{\mu\nu} \;=\; T^{\mu\nu}_{\text{Noether}} \;+\; \partial_\lambda\mathcal{K}^{[\lambda\mu]\nu}, \tag{11}$$

where $\mathcal{K}^{[\lambda\mu]\nu}$ is some 3–index Lorentz tensor antisymmetric in its first two indices.

(a) Show that regardless of the specific form of $\mathcal{K}^{[\lambda\mu]\nu}(\phi,\partial\phi)$ as a function of the fields and their derivatives,

$$\begin{aligned} \partial_\mu T^{\mu\nu} \;&=\; \partial_\mu T^{\mu\nu}_{\text{Noether}} \;=\text{(hopefully)} =\; 0 \\ P^\mu_{\text{net}} \;&\equiv\; \int\! d^3\mathbf{x}\,T^{0\mu} \;=\; \int\! d^3\mathbf{x}\,T^{0\mu}_{\text{Noether}}\,. \end{aligned} \tag{12}$$

For the scalar fields, real or complex, $T^{\mu\nu}_{\text{Noether}}$ is properly symmetric and one simply has $T^{\mu\nu} = T^{\mu\nu}_{\text{Noether}}$. Unfortunately, the situation is more complicated for the vector, tensor or spinor fields. To illustrate the problem, consider the free electromagnetic fields described by the Lagrangian

$$\mathcal{L}(A_\mu,\partial_\nu A_\mu) \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \tag{13}$$

where A_μ is a real vector field and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$.

(b) Write down $T^{\mu\nu}_{\rm Noether}$ for the free electromagnetic fields and show that it is neither symmetric nor gauge invariant.

(c) The properly symmetric — and also gauge invariant — stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\rm EM} \;=\; -F^{\mu\lambda}F^{\nu}_{\ \lambda} \;+\; \tfrac{1}{4}\,g^{\mu\nu}\,F_{\kappa\lambda}F^{\kappa\lambda}. \tag{14}$$

Show that this expression indeed has form (11) for some $\mathcal{K}^{[\lambda\mu]\nu}$.

(d) Write down the components of the stress-energy tensor (14) in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

Now consider the electromagnetic fields coupled to the electric current J^μ of some charged "matter" fields. Because of this coupling, only the *net* energy-momentum of the whole field system should be conserved, but not the separate $P^\mu_{\rm EM}$ and $P^\mu_{\rm mat}$. Consequently, we should have

$$\partial_\mu T^{\mu\nu}_{\rm net} \;=\; 0 \qquad \text{for} \quad T^{\mu\nu}_{\rm net} \;=\; T^{\mu\nu}_{\rm EM} \;+\; T^{\mu\nu}_{\rm mat} \tag{15}$$

but generally $\partial_\mu T^{\mu\nu}_{\rm EM} \neq 0$ and $\partial_\mu T^{\mu\nu}_{\rm mat} \neq 0$.

(e) Use Maxwell's equations to show that

$$\partial_\mu T^{\mu\nu}_{\rm EM} \;=\; -F^{\nu\lambda}J_\lambda \tag{16}$$

and therefore any system of charged matter fields should have its stress-energy tensor related to the electric current J_λ according to

$$\partial_\mu T^{\mu\nu}_{\rm mat} \;=\; +F^{\nu\lambda}J_\lambda. \tag{17}$$