1. Continuing the previous homework set, consider a classical theory made of a complex scalar field Φ of charge $q \neq 0$ and the the EM fields:

$$\mathcal{L}_{\mathrm{net}} \;=\; D^\mu\Phi^* D_\mu\Phi \;-\; m^2\Phi^*\Phi \;-\; \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1}$$

where

$$D_\mu\Phi \;=\; (\partial_\mu + iqA_\mu)\Phi \quad \text{and} \quad D_\mu\Phi^* \;=\; (\partial_\mu - iqA_\mu)\Phi^* \tag{2}$$

are the *covariant* derivatives.

(a) Write down the equation of motion for all fields in a covariant from. Also, write down the electric current

$$J^\mu \;\stackrel{\mathrm{def}}{=}\; -\frac{\partial\mathcal{L}}{\partial A_\mu} \tag{3}$$

in a manifestly gauge-invariant form and verify its conservation, $\partial_\mu J^\mu = 0$ (as long as the scalar fields satisfy their equations of motion).

(b) Write down the Noether stress-energy tensor for the whole field system and show that

$$T^{\mu\nu}_{\mathrm{net}} \;\equiv\; T^{\mu\nu}_{\mathrm{EM}} \;+\; T^{\mu\nu}_{\mathrm{mat}} \;=\; T^{\mu\nu}_{\mathrm{Noether}} \;+\; \partial_\lambda\mathcal{K}^{\lambda\mu\nu}, \tag{4}$$

where

$$T^{\mu\nu}_{\mathrm{EM}} \;=\; -F^{\mu\alpha}F^{\nu}_{\ \alpha} \;+\; \tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{5}$$

as for the free EM,

$$K^{\lambda\mu\nu} \;\equiv\; -K^{\mu\lambda\nu} \;=\; -F^{\lambda\mu}A^\nu, \tag{6}$$

also exactly as for the free EM, and

$$T^{\mu\nu}_{\mathrm{mat}} \;=\; D^\mu\Phi^*\, D^\nu\Phi \;+\; D^\nu\Phi^*\, D^\mu\Phi \;-\; g^{\mu\nu}\big(D_\lambda\Phi^*\, D^\lambda\Phi \;-\; m^2\Phi^*\,\Phi\big). \tag{7}$$

Note: In the presence of an electric current J^μ, the $\partial_\lambda\mathcal{K}^{\lambda\mu\nu}$ correction to the electromagnetic stress-energy tensor contains an extra $J^\mu A^\nu$ term. This term is important for obtaining a gauge-invariant stress-energy tensor (7) for the scalar field.

(c) Use the scalar fields' equations of motion and the non-commutativity of covariant derivatives

$$[D_\mu, D_\nu]\Phi \;=\; iqF_{\mu\nu}\Phi, \qquad [D_\mu, D_\nu]\Phi^* \;=\; -iqF_{\mu\nu}\Phi^* \tag{8}$$

to show that

$$\partial_\mu T^{\mu\nu}_{\text{mat}} \;=\; +F^{\nu\lambda}J_\lambda \tag{9}$$

and therefore the *net* stress-energy tensor (4) is conserved.

Note: the last statement follows from problem 1.3 (e). Do not redo that problem here, just quote the result.

2. In class, we discussed field multiplets $\Psi_i(x)$ which transform as (complex) vectors under the $SU(N)$ symmetry,

$$\Psi'(x) \;=\; U(x)\Psi(x) \quad i.e. \quad \Psi'_i(x) \;=\sum_j U_i^{\ j}(x)\Psi_j(x), \quad i,j=1,2,\ldots,N \tag{10}$$

where $U(x)$ is an x–dependent unitary $N\times N$ matrix, $\det U(x)\equiv 1$. Now consider the *adjoint multiplet* $\Phi_i^{\ j}(x)$ of fields: for each x, it comprises a traceless hermitian $N\times N$ matrix $\Phi(x)$ which transforms according to

$$\Phi'(x) \;=\; U(x)\Phi(x)U^\dagger(x), \quad i.e. \quad \Phi_i'^{j}(x) \;=\sum_{k,\ell} U_i^{\ k}(x)\,\Phi_k^{\ \ell}(x)\,U_\ell^{\dagger j}(x). \tag{11}$$

Note that this transformation law preserves the $\Phi^\dagger=\Phi$ and $\mathrm{tr}(\Phi)=0$ conditions.

The covariant derivative acts on the adjoint multiplet according to

$$D_\mu\Phi(x) \;=\; \partial_\mu\Phi(x) \;+\; i[A_\mu(x),\Phi(x)] \;\equiv\; \partial_\mu(x) \;+\; iA_\mu(x)\Phi(x) \;-\; i\Phi(x)A_\mu(x) \tag{12}$$

(a) Verify that this derivative is indeed covariant and $D_\mu\Phi(x)$ transforms under the local $SU(N)$ symmetry exactly like $\Phi(x)$ itself.

(b) Show that $[D_\mu, D_\nu]\Phi(x) = i[F_{\mu\nu}(x),\Phi(x)]$.

The non-abelian tension field $F_{\mu\nu}(x)$ itself transforms according to the adjoint representation of the local symmetry, $F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^\dagger(x)$. Hence, the covariant derivative acts on the tension field according to $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + i[A_\lambda, F_{\mu\nu}]$.

(c) Verify the non-abelian Bianchi identity $D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0$.

(d) Show that for an infinitesimal variation of the non-abelian gauge field $A_\nu(x) \to A_\nu(x) + \delta A_\nu(x)$, the tension varies according to $\delta F_{\mu\nu}(x) = D_\mu \delta A_\nu(x) - D_\nu \delta A_\mu(x)$.

(e) *Note: This question was changed Friday 9/12 at 22:30.*

The Yang–Mills theory is has a non-abelian gauge symmetry, but its only fields are the gauge fields $A_i^{\mu j}(x) = \sum_a (\frac{\lambda^a}{2})_i^j \times A_\mu^a(x)$ themselves; there are no other fields. The Yang–Mills Lagrangian is

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \;=\; \sum_a \frac{-1}{4g^2}\,F^a_{\mu\nu}F^{a\mu\nu}. \tag{13}$$

Write down the classical equations of motion for this theory.