1. Consider a non-abelian gauge theory comprising N complex scalar fields $\phi_i(x)$ and N^2-1 real vector fields $A^a_\mu(x)$. In matrix notations, the Lagrangian of the theory is

$$\mathcal{L} \;=\; -\frac{1}{2g^2}\,\mathrm{tr}\big(F^{\mu\nu}F_{\mu\nu}\big) \;+\; D_\mu\Phi^\dagger\,D^\mu\Phi \;-\; m^2\,\Phi^\dagger\Phi. \tag{1}$$

   (a) Derive classical equations of motion for all the fields and write those equation in a covariant form. In particular, show that the vector fields satisfy

$$D_\mu F^{\mu\nu}(x) \;=\; g^2\,J^\nu(x) \;\equiv\; g^2\sum_a \frac{\lambda^a}{2}\times J^{a\nu}(x) \tag{2}$$

   where the currents $J^{a\mu}(x)$ involve the scalar fields and their covariant derivatives.

   (b) Show that regardless of the specific form of the currents $J^{a\mu}(x)$, eqs. (2) require those currents to be *covariantly conserved*, $D_\mu J^\mu(x)=0$.

   (c) Show that when the scalar fields $\Phi(x)$ and $\Phi^\dagger(x)$ satisfy their equations of motion, the currents $J^\mu(x)$ are indeed covariantly conserved.

   Note that *covariantly conserved* currents $J^{a\mu}(x)$ do *not* give rise to conserved charges $Q^a=\int d^3\mathbf{x}\,J^{a0}(\mathbf{x},t)$. That's one more reason why the non-abelian gauge theories are much more complicated than the electromagnetism.

2. Consider a *massive* relativistic vector field $A^\mu(x)$ with the Lagrangian density

$$\mathcal{L} \;=\; -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} \;+\; \tfrac{1}{2}m^2\,A_\mu A^\mu \;-\; A^\mu J_\mu \tag{3}$$

   (in $\hbar=c=1$ units) where the current $J^\mu(x)$ is a fixed source for the $A^\mu(x)$ field. Because of the mass term, the Lagrangian (3) is *not* gauge invariant. However, we *assume* that the current $J^\mu(x)$ is conserved, $\partial_\mu J^\mu(x)=0$.

In an earlier homework (set 1, problem 1) we have derived the Euler–Lagrange equations for the massive vector field. In this problem, we develop the Hamiltonian formalism for the $A^\mu(x)$. Our first step is to identify the canonically conjugate "momentum" fields.

(a) Show that $\partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\mathbf{E}$ but $\partial\mathcal{L}/\partial\dot{A}_0 \equiv 0$.

In other words, the canonically conjugate field to $\mathbf{A}(\mathbf{x})$ is $-\mathbf{E}(\mathbf{x})$ but the $A_0(\mathbf{x})$ does not have a canonical conjugate! Consequently,

$$H \;=\; \int\! d^3\mathbf{x}\left(-\dot{\mathbf{A}}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) \;-\; \mathcal{L}\right). \tag{4}$$

(b) Show that in terms of the $\mathbf{A}$, $\mathbf{E}$, and A_0 fields, and their *space* derivatives,

$$H = \int\! d^3\mathbf{x}\left\{\tfrac{1}{2}\mathbf{E}^2 \,+\, A_0\left(J_0 - \nabla\cdot\mathbf{E}\right) \,-\, \tfrac{1}{2}m^2A_0^2 \,+\, \tfrac{1}{2}\left(\nabla\times\mathbf{A}\right)^2 \,+\, \tfrac{1}{2}m^2\mathbf{A}^2 \,-\, \mathbf{J}\cdot\mathbf{A}\right\}. \tag{5}$$

Because the A_0 field does not have a canonical conjugate, the Hamiltonian formalism does not produce an equation for the time-dependence of this field. Instead, it gives us a time-independent equation relating the $A_0(\mathbf{x},t)$ to the values of other fields *at the same time* t. Specifically, we have

$$\frac{\delta H}{\delta A_0(\mathbf{x})} \;\equiv\; \left.\frac{\partial\mathcal{H}}{\partial A_0}\right|_{\mathbf{x}} \;-\; \nabla\cdot\left.\frac{\partial\mathcal{H}}{\partial(\nabla A_0)}\right|_{\mathbf{x}} \;=\; 0. \tag{6}$$

At the same time, the vector fields $\mathbf{A}$ and $\mathbf{E}$ satisfy the Hamiltonian equations of motion,

$$\begin{aligned}
\frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t) \;&=\; -\left.\frac{\delta H}{\delta\mathbf{E}(\mathbf{x})}\right|_t \;\equiv\; -\left[\frac{\partial\mathcal{H}}{\partial\mathbf{E}} \,-\, \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{E}}\right]_{(\mathbf{x},t)},\\
\frac{\partial}{\partial t}\mathbf{E}(\mathbf{x},t) \;&=\; +\left.\frac{\delta H}{\delta\mathbf{A}(\mathbf{x})}\right|_t \;\equiv\; +\left[\frac{\partial\mathcal{H}}{\partial\mathbf{A}} \,-\, \nabla_i\frac{\partial\mathcal{H}}{\partial(\nabla_i\mathbf{A}}\right]_{(\mathbf{x},t)}.
\end{aligned} \tag{7}$$

(c) Write down the explicit form of all these equations.

(d) Verify that the equations you have just written down are equivalent to the relativistic Euler–Lagrange equations for the $A^\mu(x)$, namely

$$(\partial^\mu\partial_\mu + m^2)A^\nu \;=\; \partial^\nu(\partial_\mu A^\mu) \;+\; J^\nu \tag{8}$$

and hence $\partial_\mu A^\mu(x) = 0$ and $(\partial^\nu\partial_\nu + m^2)A^\mu = 0$ when $\partial_\mu J^\mu \equiv 0$, *cf.* homework #1.

3. Finally, let's quantize the massive vector fields. Since classically the $-\mathbf{E}(\mathbf{x})$ fields are canonically conjugate momenta to the $\mathbf{A}(\mathbf{x})$ fields, the corresponding quantum fields $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ satisfy the canonical equal-time commutation relations

$$\begin{aligned}
[\hat{A}_i(\mathbf{x},t), \hat{A}_j(\mathbf{y},t)] &= 0, \\
[\hat{E}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= 0, \\
[\hat{A}_i(\mathbf{x},t), \hat{E}_j(\mathbf{y},t)] &= -i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})
\end{aligned} \tag{9}$$

(in the $\hbar = c = 1$ units). The currents also become quantum fields $\hat{J}^\mu(\mathbf{x},t)$, but they are composed of some kind of charged degrees of freedom rather than the vector fields in question. Consequently, the $\hat{J}^\mu(\mathbf{x},t)$ commute with both $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\mathbf{A}}(\mathbf{x})$ fields.

The classical $A^0(\mathbf{x},t)$ field does not have a canonical conjugate and its equation of motion does not involve time derivatives. In the quantum theory, $\hat{A}^0(\mathbf{x},t)$ satisfies a similar time-independent constraint

$$m^2\hat{A}^0(\mathbf{x},t) = \hat{J}^0(\mathbf{x},t) - \nabla\cdot\hat{\mathbf{E}}(\mathbf{x},t). \tag{10}$$

From the Hilbert space point of view, this is an operatorial identity rather than an equation of motion. Consequently, the commutation relations of the $\hat{A}^0(\mathbf{x},t)$ field follow from eqs. (9); in particular, $\hat{A}^0(\mathbf{x},t)$ commutes with the $\hat{\mathbf{E}}(\mathbf{x},t)$ but does not commute with the $\hat{\mathbf{A}}(\mathbf{x},t)$.

Finally, the Hamiltonian operator follows from the classical eq. (5), namely

$$\begin{aligned}
\hat{H} &= \int\! d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \hat{A}_0\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right) - \tfrac{1}{2}m^2\hat{A}_0^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right\} \\
&= \int\! d^3\mathbf{x} \left\{ \tfrac{1}{2}\hat{\mathbf{E}}^2 + \frac{1}{2m^2}\left(\hat{J}_0 - \nabla\cdot\hat{\mathbf{E}}\right)^2 + \tfrac{1}{2}\left(\nabla\times\hat{\mathbf{A}}\right)^2 + \tfrac{1}{2}m^2\hat{\mathbf{A}}^2 - \hat{\mathbf{J}}\cdot\hat{\mathbf{A}} \right\}
\end{aligned} \tag{11}$$

where the second line follows from the first and eq. (10).

Your task is to calculate the commutators $[\hat{A}_i(\mathbf{x},t), \hat{H}]$ and $[\hat{E}_i(\mathbf{x},t), \hat{H}]$ and write down the Heisenberg equations for the quantum vector fields. Make sure those equations are similar to the Hamilton equations for the classical fields.