

1. An operator acting on identical bosons can be described in terms of N -particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

The key to this conversion are the single-particle wave functions $\phi_\alpha(\mathbf{x})$ of states $|\alpha\rangle$ and the *symmetrized* N -particle states wave functions

$$\begin{aligned} \phi_{\alpha\beta\dots\omega}(\mathbf{x}_1, \mathbf{x}_2 \dots, \mathbf{x}_N) &= \frac{1}{\sqrt{D}} \sum_{\substack{\text{distinct permutations} \\ \text{of } (\alpha, \beta, \dots, \omega) \\ (\tilde{\alpha}, \tilde{\beta}, \dots, \tilde{\omega})}} \phi_{\tilde{\alpha}}(\mathbf{x}_1) \times \phi_{\tilde{\beta}}(\mathbf{x}_2) \times \dots \times \phi_{\tilde{\omega}}(\mathbf{x}_N) \\ &= \frac{1}{T\sqrt{D}} \sum_{\substack{\text{all permutations} \\ \text{of } (\alpha, \beta, \dots, \omega) \\ (\tilde{\alpha}, \tilde{\beta}, \dots, \tilde{\omega})}} \phi_{\tilde{\alpha}}(\mathbf{x}_1) \times \phi_{\tilde{\beta}}(\mathbf{x}_2) \times \dots \times \phi_{\tilde{\omega}}(\mathbf{x}_N) \end{aligned} \quad (1)$$

of N -boson states $|\alpha, \beta, \dots, \omega\rangle$. In eqs. (1), D is the number of *distinct* (*i.e.*, non-trivial) permutations of single-particle states $(\alpha, \beta, \dots, \omega)$ and T is the number of trivial permutations. In terms of the occupation numbers n_γ

$$T = \prod_{\gamma} n_{\gamma}!, \quad D = \frac{N!}{T}. \quad (2)$$

- (a) Consider a generic N -particle quantum state $|N; \psi\rangle$ with a totally symmetric wave-function $\Psi(\mathbf{x}_1, \dots, \mathbf{x}_N)$. Show that the $(N+1)$ -particle state $|N+1, \psi'\rangle = \hat{a}_\alpha^\dagger |N; \psi\rangle$ has wave function

$$\psi'(\mathbf{x}_1, \dots, \mathbf{x}_N) = \frac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \phi_\alpha(\mathbf{x}_i) \times \psi(\mathbf{x}_1, \dots, \cancel{\mathbf{x}_i}, \dots, \mathbf{x}_{N+1}). \quad (3)$$

Hint: First prove this for wave-functions of the form (1). Then use the fact that states $|\alpha_1, \dots, \alpha_N\rangle$ form a complete basis of the N -boson Hilbert space.

(b) Show that the $(N - 1)$ -particle state $|N - 1, \psi''\rangle = \hat{a}_\alpha |N, \psi\rangle$ has wave-function

$$\psi''(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}) = \sqrt{N} \int d^3\mathbf{x}_N \phi_\alpha^*(\mathbf{x}_N) \times \psi(\mathbf{x}_1, \dots, \mathbf{x}_{N-1}, \mathbf{x}_N). \quad (4)$$

Hint: for any $|N - 1, \tilde{\psi}\rangle$, $\langle N - 1, \tilde{\psi} | \hat{a}_\alpha |N, \psi\rangle = \langle N, \psi | \hat{a}_\alpha^\dagger |N - 1, \tilde{\psi}\rangle^*$.

Now consider one-body operators, *i.e.* additive operators acting on one particle at a time. In the first-quantized formalism they act on N -particle states according to

$$\hat{A}_{\text{net}}^{(1)} = \sum_{i=1}^N \hat{A}_1(i^{\text{th}} \text{ particle}) \quad (5)$$

where \hat{A}_1 is some kind of a one-particle operator (such as momentum $\hat{\mathbf{p}}$, or kinetic energy $\frac{1}{2m}\hat{\mathbf{p}}^2$, or potential $V(\hat{\mathbf{x}})$, *etc., etc.*). In the second-quantized formalism such operators become

$$\hat{A}_{\text{net}}^{(2)} = \sum_{\alpha, \beta} \langle \alpha | \hat{A}_1 | \beta \rangle \hat{a}_\alpha^\dagger \hat{a}_\beta. \quad (6)$$

(c) Verify that the two operators have the same matrix elements between any two N -boson states $|N, \psi\rangle$ and $|N, \tilde{\psi}\rangle$, $\langle N, \tilde{\psi} | \hat{A}_{\text{net}}^{(1)} |N, \psi\rangle = \langle N, \tilde{\psi} | \hat{A}_{\text{net}}^{(2)} |N, \psi\rangle$.

Hint: use $\hat{A}_1 = \sum_{\alpha, \beta} |\alpha\rangle \langle \alpha | \hat{A}_1 | \beta \rangle \langle \beta |$.

Finally, consider two-body operators, *i.e.* additive operators acting on two particles at a time. Given a two-particle operator \hat{B}_2 — such as $V(\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2)$ — the *net* B operator acts in the first-quantized formalism according to

$$\hat{B}_{\text{net}}^{(1)} = \frac{1}{2} \sum_{i \neq j} \hat{B}_2(i^{\text{th}} \text{ and } j^{\text{th}} \text{ particles}), \quad (7)$$

and in the second-quantized formalism according to

$$\hat{B}_{\text{net}}^{(2)} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} (\langle \alpha | \otimes \langle \beta |) \hat{B}_2(|\gamma\rangle \otimes |\delta\rangle) \hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta. \quad (8)$$

(d) Again, show these two operators have the same matrix elements between any two N -boson states, $\langle N, \tilde{\psi} | \hat{A}_{\text{net}}^{(1)} |N, \psi\rangle = \langle N, \tilde{\psi} | \hat{A}_{\text{net}}^{(2)} |N, \psi\rangle$ for any $\langle N, \tilde{\psi} |$ and $|N, \psi\rangle$.

2. Next, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] = 0, \quad [\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger] = 0, \quad [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}. \quad (9)$$

- (a) Calculate the commutators $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger]$, $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}_\alpha^\dagger \hat{a}_\beta, \hat{a}_\gamma^\dagger \hat{a}_\delta]$.
- (b) Consider three one-particle operators \hat{A}_1 , \hat{B}_1 , and \hat{C}_1 . Let us define the corresponding second-quantized operators $\hat{A}_{\text{net}}^{(2)}$, $\hat{B}_{\text{net}}^{(2)}$, and $\hat{C}_{\text{net}}^{(2)}$ according to eq. (6).

$$\text{Show that if } \hat{C}_1 = [\hat{A}_1, \hat{B}_1] \text{ then } \hat{C}_{\text{net}}^{(2)} = [\hat{A}_{\text{net}}^{(2)}, \hat{B}_{\text{net}}^{(2)}].$$

- (c) Next, calculate the commutator $[\hat{a}_\alpha^\dagger \hat{a}_\beta^\dagger \hat{a}_\gamma \hat{a}_\delta, \hat{a}_\mu^\dagger \hat{a}_\nu]$.
- (d) Finally, let \hat{A}_1 be a one-particle operator, let \hat{B}_2 and \hat{C}_2 be two-body operators, and let $\hat{A}_{\text{net}}^{(2)}$, $\hat{B}_{\text{net}}^{(2)}$, and $\hat{C}_{\text{net}}^{(2)}$ be the corresponding second-quantized operators according to eqs. (6) and (8).

$$\text{Show that if } \hat{C}_2 = \left[\left(\hat{A}_1^{(1^{\text{st}})} + \hat{A}_1^{(2^{\text{nd}})} \right), \hat{B}_2 \right] \text{ then } \hat{C}_{\text{net}}^{(2)} = \left[\hat{A}_{\text{net}}^{(2)}, \hat{B}_{\text{net}}^{(2)} \right].$$

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$.

- (a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\text{def}}{=} \exp(\xi \hat{a}^\dagger - \xi^* \hat{a}) |0\rangle$. Show that

$$|\xi\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger} |0\rangle \quad \text{and} \quad \hat{a} |\xi\rangle = \xi |\xi\rangle. \quad (10)$$

- (b) Use $\hat{a} |\xi\rangle = \xi |\xi\rangle$ to show that the (coordinate-space) wave function of a coherent state $|\xi\rangle$ is a gaussian wave packet of the same width as the ground state $|0\rangle$.
- (c) Use $\hat{a} |\xi\rangle = \xi |\xi\rangle$ and $\langle \xi | \hat{a}^\dagger = \xi^* \langle \xi |$ to calculate the uncertainties Δq and Δp in a coherent state and verify their minimality: $\Delta q \Delta p = \frac{1}{2} \hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\text{def}}{=} \langle \hat{n} \rangle = |\xi|^2$.
- (d) Show that the (coordinate-space) wave function of a coherent state $|\xi\rangle$ is a gaussian wave packet of the same width as the ground state $|0\rangle$.

- (e) Consider time-dependent coherent states $|\xi(t)\rangle$. Show that for $\xi(t) = \xi_0 e^{-i\omega t}$, the state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar \frac{d}{dt} |\xi(t)\rangle = \hat{H} |\xi(t)\rangle$.
- (f) The coherent states are not quite orthogonal to each other.
Calculate their overlap $\langle \eta | \xi \rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

- (g) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x}) |\Phi\rangle = \Phi(\mathbf{x}) |\Phi\rangle \quad (11)$$

for any given classical complex field $\Phi(\mathbf{x})$.

- (h) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \stackrel{\text{def}}{=} \langle \Phi | \hat{N} | \Phi \rangle = \int d\mathbf{x} |\Phi(\mathbf{x})|^2. \quad (12)$$

- (i) Let

$$\hat{H} = \int d\mathbf{x} \left(\frac{\hbar^2}{2M} \nabla \hat{\Psi}^\dagger(\mathbf{x}) \cdot \nabla \hat{\Psi}(\mathbf{x}) + V(\mathbf{x}) \hat{\Psi}^\dagger(\mathbf{x}) \hat{\Psi}(\mathbf{x}) \right)$$

and show that for any classical field configuration $\Phi(\mathbf{x}, t)$ that satisfies the classical field equation

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{x}, t) = \left(-\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \Phi(\mathbf{x}, t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Phi\rangle = \hat{H} |\Phi\rangle. \quad (13)$$

- (j) Finally, show that the quantum overlap $|\langle \Phi_1 | \Phi_2 \rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.