1. An operator acting on identical bosons can be described in terms of N–particle wave functions (the *first-quantized* formalism) or in terms of creation and annihilation operators in the Fock space (the *second-quantized* formalism). This exercise is about converting the operators from one formalism to another.

   The key to this conversion are the single-particle wave functions $\phi_\alpha(\mathbf{x})$ of states $|\alpha\rangle$ and the *symmetrized* N-particle states wave functions

$$\begin{aligned}\phi_{\alpha\beta\cdots\omega}(\mathbf{x}_1,\mathbf{x}_2\,\ldots,\mathbf{x}_N) &= \frac{1}{\sqrt{D}}\sum_{(\tilde\alpha,\tilde\beta,\ldots,\tilde\omega)}^{\substack{\text{distinct permutations}\\ \text{of }(\alpha,\beta,\ldots,\omega)}} \phi_{\tilde\alpha}(\mathbf{x}_1)\times\phi_{\tilde\beta}(\mathbf{x}_2)\times\cdots\times\phi_{\tilde\omega}(\mathbf{x}_N)\\ &= \frac{1}{T\sqrt{D}}\sum_{(\tilde\alpha,\tilde\beta,\ldots,\tilde\omega)}^{\substack{\text{all permutations}\\ \text{of }(\alpha,\beta,\ldots,\omega)}} \phi_{\tilde\alpha}(\mathbf{x}_1)\times\phi_{\tilde\beta}(\mathbf{x}_2)\times\cdots\times\phi_{\tilde\omega}(\mathbf{x}_N)\end{aligned}\tag{1}$$

   of N-boson states $|\alpha,\beta,\ldots,\omega\rangle$. In eqs. (1), D is the number of *distinct* (*i.e.*, non-trivial) permutations of single-particle states $(\alpha,\beta,\ldots,\omega)$ and T is the number of trivial permutations. In terms of the occupation numbers n_γ

$$T = \prod_\gamma n_\gamma!\,,\qquad D = \frac{N!}{T}\,.\tag{2}$$

   (a) Consider a generic N-particle quantum state $|N;\psi)\rangle$ with a totally symmetric wavefunction $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$. Show that the $(N+1)$–particle state $|N+1,\psi'\rangle = \hat{a}^\dagger_\alpha\,|N;\psi\rangle$ has wave function

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_N) = \frac{1}{\sqrt{N+1}}\sum_{i=1}^{N+1}\phi_\alpha(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\not{\mathbf{x}}_i,\ldots,\mathbf{x}_{N+1}).\tag{3}$$

   Hint: First prove this for wave-functions of the form (1). Then use the fact that states $|\alpha_1,\ldots,\alpha_N\rangle$ form a complete basis of the N-boson Hilbert space.

(b) Show that the $(N-1)$–particle state $|N-1,\psi''\rangle = \hat{a}_\alpha\,|N;\psi\rangle$ has wave-function

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int\! d^3\mathbf{x}_N\,\phi^*_\alpha(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{4}$$

Hint: for any $|N-1,\widetilde{\psi}\rangle$, $\langle N-1,\widetilde{\psi}|\,\hat{a}_\alpha\,|N,\psi\rangle = \langle N,\psi|\,\hat{a}^\dagger_\alpha\,|N-1,\widetilde{\psi}\rangle^*$.

Now consider one-body operators, *i.e.* additive operators acting on one particle at a time. In the first-quantized formalism they act on N–particle states according to

$$\hat{A}^{(1)}_{\rm net} \;=\; \sum_{i=1}^{N}\hat{A}_1(i^{\rm th}\ \text{particle}) \tag{5}$$

where $\hat{A}_1$ is some kind of a one-particle operator (such as momentum $\hat{\mathbf{p}}$, or kinetic energy $\frac{1}{2m}\hat{\mathbf{p}}^2$, or potential $V(\hat{\mathbf{x}})$, *etc.*, *etc.*). In the second-quantized formalism such operators become

$$\hat{A}^{(2)}_{\rm net} \;=\; \sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\;\hat{a}^\dagger_\alpha\hat{a}_\beta\,. \tag{6}$$

(c) Verify that the two operators have the same matrix elements between any two N-boson states $|N,\psi\rangle$ and $|N,\widetilde{\psi}\rangle$, $\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle = \langle N,\widetilde{\psi}|\,\hat{A}^{(2)}_{\rm net}\,|N,\psi\rangle$.
Hint: use $\hat{A}_1 = \sum_{\alpha,\beta}|\alpha\rangle\,\langle\alpha|\,\hat{A}_1\,|\beta\rangle\,\langle\beta|$.

Finally, consider two-body operators, *i.e.* additive operators acting on two particles at a time. Given a two-particle operator $\hat{B}_2$ — such as $V(\hat{\mathbf{x}}_1-\hat{\mathbf{x}}_2)$ — the *net* B operator acts in the first-quantized formalism according to

$$\hat{B}^{(1)}_{\rm net} \;=\; \tfrac{1}{2}\sum_{i\neq j}\hat{B}_2(i^{\rm th}\ \text{and}\ j^{\rm th}\ \text{particles}), \tag{7}$$

and in the second-quantized formalism according to

$$\hat{B}^{(2)}_{\rm net} \;=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta}\big(\langle\alpha|\otimes\langle\beta|\big)\hat{B}_2\big(|\gamma\rangle\otimes|\delta\rangle\big)\,\hat{a}^\dagger_\alpha\hat{a}^\dagger_\beta\hat{a}_\gamma\hat{a}_\delta\,. \tag{8}$$

(d) Again, show these two operators have the same matrix elements between any two N-boson states, $\langle N,\widetilde{\psi}|\,\hat{A}^{(1)}_{\rm net}\,|N,\psi\rangle = \langle N,\widetilde{\psi}|\,\hat{A}^{(2)}_{\rm net}\,|N,\psi\rangle$ for any $\langle N,\widetilde{\psi}|$ and $|N,\psi\rangle$.

2. Next, an exercise in bosonic commutation relations

$$[\hat{a}_\alpha, \hat{a}_\beta] \;=\; 0, \qquad [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] \;=\; 0, \qquad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta}\,. \tag{9}$$

(a) Calculate the commutators $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma]$, $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}_\delta]$ and $[\hat{a}^\dagger_\alpha \hat{a}_\beta, \hat{a}^\dagger_\gamma \hat{a}_\delta]$.

(b) Consider three one-particle operators $\hat{A}_1$, $\hat{B}_1$, and $\hat{C}_1$. Let us define the corresponding second-quantized operators $\hat{A}^{(2)}_{\rm net}$, $\hat{B}^{(2)}_{\rm net}$, and $\hat{C}^{(2)}_{\rm net}$ according to eq. (6).

Show that if $\hat{C}_1 = [\hat{A}_1, \hat{B}_1]$ then $\hat{C}^{(2)}_{\rm net} = \left[\hat{A}^{(2)}_{\rm net}, \hat{B}^{(2)}_{\rm net}\right]$.

(c) Next, calculate the commutator $[\hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\gamma \hat{a}_\delta, \hat{a}^\dagger_\mu \hat{a}_\nu]$.

(d) Finally, let $\hat{A}_1$ be a one-particle operator, let $\hat{B}_2$ and $\hat{C}_2$ be two-body operators, and let $\hat{A}^{(2)}_{\rm net}$, $\hat{B}^{(2)}_{\rm net}$, and $\hat{C}^{(2)}_{\rm net}$ be the corresponding second-quantized operators according to eqs. (6) and (8).

Show that if $\hat{C}_2 = \left[\left(\hat{A}_1(1^{\rm st}) + \hat{A}_1(2^{\rm nd})\right), \hat{B}_2\right]$ then $\hat{C}^{(2)}_{\rm net} = \left[\hat{A}^{(2)}_{\rm net}, \hat{B}^{(2)}_{\rm net}\right]$.

3. The rest of this homework is about coherent states of harmonic oscillators and free quantum fields. Let us start with a harmonic oscillator $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$.

(a) For any complex number ξ we define a *coherent state* $|\xi\rangle \stackrel{\rm def}{=} \exp\bigl(\xi\hat{a}^\dagger - \xi^*\hat{a}\bigr)\,|0\rangle$. Show that

$$|\xi\rangle \;=\; e^{-|\xi|^2/2}\, e^{\xi\hat{a}^\dagger}\,|0\rangle \quad \text{and} \quad \hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle\,. \tag{10}$$

(b) Use $\hat{a}\,|\xi\rangle \;=\; \xi\,|\xi\rangle$ to show that the (coordinate-space) wave function of a coherent state $|\xi\rangle$ is a gaussian wave packet of the same width as the ground state $|0\rangle$.

(c) Use use $\hat{a}\,|\xi\rangle = \xi\,|\xi\rangle$ and $\langle\xi|\,\hat{a}^\dagger = \xi^*\,\langle\xi|$ to calculate the uncertainties Δq and Δp in a coherent state and verify their minimality: $\Delta q \Delta p = \frac{1}{2}\hbar$. Also, verify $\delta n = \sqrt{\bar{n}}$ where $\bar{n} \stackrel{\rm def}{=} \langle\hat{n}\rangle = |\xi|^2$.

(d) Show that the (coordinate-space) wave function of a coherent state $|\xi\rangle$ is a gaussian wave packet of the same width as the ground state $|0\rangle$.

(e) Consider time-dependent coherent states $|\xi(t)\rangle$. Show that for $\xi(t) = \xi_0 e^{-i\omega t}$, the state $|\xi(t)\rangle$ satisfies the time-dependent Schrödinger equation $i\hbar\frac{d}{dt}|\xi(t)\rangle = \hat{H}|\xi(t)\rangle$.

(f) The coherent states are not quite orthogonal to each other.
Calculate their overlap $\langle\eta|\xi\rangle$.

Now consider coherent states of multi-oscillator systems and hence quantum fields. In particular, let us focus on the creation and annihilation fields $\hat{\Psi}^\dagger(\mathbf{x})$ and $\hat{\Psi}(\mathbf{x})$ for non-relativistic spinless bosons.

(g) Generalize (a) and construct coherent states $|\Phi\rangle$ which satisfy

$$\hat{\Psi}(\mathbf{x})\,|\Phi\rangle \;=\; \Phi(\mathbf{x})\,|\Phi\rangle \tag{11}$$

for any given classical complex field $\Phi(\mathbf{x})$.

(h) Show that for any such coherent state, $\Delta N = \sqrt{\bar{N}}$ where

$$\bar{N} \;\stackrel{\text{def}}{=}\; \langle\Phi|\,\hat{N}\,|\Phi\rangle \;=\; \int d\mathbf{x}\,|\Phi(\mathbf{x})|^2. \tag{12}$$

(i) Let

$$\hat{H} \;=\; \int d\mathbf{x}\left(\frac{\hbar^2}{2M}\nabla\hat{\Psi}^\dagger(\mathbf{x})\cdot\nabla\hat{\Psi}(\mathbf{x}) \;+\; V(\mathbf{x})\times\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{x})\right)$$

and show that for any classical field configuration $\Phi(\mathbf{x},t)$ that satisfies the classical field equation

$$i\hbar\frac{\partial}{\partial t}\Phi(\mathbf{x},t) \;=\; \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\mathbf{x})\right)\Phi(\mathbf{x},t),$$

the time-dependent coherent state $|\Phi\rangle$ satisfies the true Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\Phi\rangle \;=\; \hat{H}\,|\Phi\rangle\,. \tag{13}$$

(j) Finally, show that the quantum overlap $|\langle\Phi_1|\Phi_2\rangle|^2$ between two different coherent states is exponentially small for any *macroscopic* difference $\delta\Phi(\mathbf{x}) = \Phi_1(\mathbf{x}) - \Phi_2(\mathbf{x})$ between the two field configurations.