1. Let's start with the Bogolyubov transform. Given some kind of annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ which satisfy the bosonic commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0, \qquad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}\,, \tag{1}$$

we define new operators $\hat{b}_{\mathbf{k}}$ and $\hat{b}^\dagger_{\mathbf{k}}$ according to

$$\hat{b}_{\mathbf{k}} = \cosh(t_{\mathbf{k}})\hat{a}_{\mathbf{k}} + \sinh(t_{\mathbf{k}})\hat{a}^\dagger_{-\mathbf{k}}\,, \quad \hat{b}^\dagger_{\mathbf{k}} = \cosh(t_{\mathbf{k}})\hat{a}^\dagger_{\mathbf{k}} + \sinh(t_{\mathbf{k}})\hat{a}_{-\mathbf{k}} \tag{2}$$

for some arbitrary real parameters $t_{\mathbf{k}} = t_{-\mathbf{k}}$.

(a) Show that the $\hat{b}_{\mathbf{k}}$ and the $\hat{b}^\dagger_{\mathbf{k}}$ satisfy the same bosonic commutation relations as the $\hat{a}_{\mathbf{k}}$ and the $\hat{a}^\dagger_{\mathbf{k}}$.

The Bogolyubov transform — replacing the 'original' creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ with the 'transformed' operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ — is useful for diagonalizing quadratic Hamiltonians of the form

$$\hat{H} = \sum_{\mathbf{k}} A_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \tfrac{1}{2}\sum_{\mathbf{k}} B_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\right) \tag{3}$$

where for all momenta $\mathbf{k}$, $A_{\mathbf{k}} = A_{-\mathbf{k}}$, $B_{\mathbf{k}} = B_{-\mathbf{k}}$, and $A_{\mathbf{k}} > |B_{\mathbf{k}}|$.

(b) Show that for a suitable choice of the $t_{\mathbf{k}}$ parameters,

$$\hat{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} + \text{const} \qquad \text{where } \omega_{\mathbf{k}} = \sqrt{A^2_{\mathbf{k}} - B^2_{\mathbf{k}}}\,. \tag{4}$$

(c) Show that $\hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}} - \hat{b}^\dagger_{-\mathbf{k}}\hat{b}_{-\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{-\mathbf{k}}\hat{a}_{-\mathbf{k}}$ and therefore

$$\hat{\mathbf{P}} \equiv \sum_{\mathbf{k}} \mathbf{k} \times \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} = \sum_{\mathbf{k}} \mathbf{k} \times \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}\,. \tag{5}$$

2. Now consider the quantum field theory of superfluid helium. In class we have shifted the quantum fields by $\hat{\Psi}(x) = \sqrt{n} + \delta\hat{\Psi}(x)$, expanded the Hamiltonian in powers of the $\delta\hat{\Psi}(x)$ and $\delta\hat{\Psi}^\dagger(x)$ as $\hat{H} - \mu\hat{N} = (E - \mu N) + \hat{H}_2 + \hat{H}_{3+4}$ where $\hat{H}_2$ describes free quasiparticles and $\hat{H}_{3+4}$ their interactions, wrote $\hat{H}_2$ in terms of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$, and finally used a Bogolyubov transform (4) to obtain

$$\hat{H}_2 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}} + \text{const} \tag{6}$$

where

$$\omega_{\mathbf{k}} = |\mathbf{k}| \times \sqrt{\frac{\lambda n}{M} + \frac{\mathbf{k}^2}{4M^2}}. \tag{7}$$

All that was done for a zero-range force between two helium atoms, $V_2(\mathbf{x}-\mathbf{y}) = \lambda\delta^{(3)}(\mathbf{x}-\mathbf{y})$.

Now, let us allow for a more general model of liquid helium in which the two-body forces between the atoms have finite range. In QFT terms, this gives us a non-local free-energy operator

$$\begin{aligned}\hat{F} \equiv \hat{H} - \mu\hat{N} = &\int d^3\mathbf{x} \left(\frac{1}{2M} \nabla\hat{\Psi}^\dagger(x) \cdot \nabla\hat{\Psi}(\mathbf{x}) - \mu\hat{\Psi}^\dagger(x)\hat{\Psi}(\mathbf{x}) \right) \\ &+ \tfrac{1}{2} \iint d^3\mathbf{x}\, d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{y})\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y}).\end{aligned} \tag{8}$$

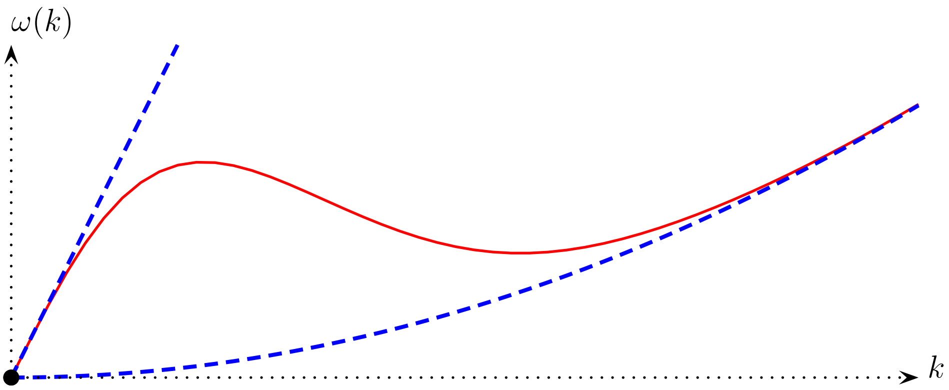
(a) Find the density $n = N/L^3$ which minimizes the free energy, shift the quantum fields by $\left\langle \hat{\Psi} \right\rangle = \sqrt{n}$, expand $\hat{F}$ in powers of the shifted fields as $\hat{F} = (E - \mu N) + \hat{H}_2 + \hat{H}_{3+4}$, and show that

$$\begin{aligned}\hat{H}_2 = &\int d^3\mathbf{x} \frac{1}{2M} \nabla\hat{\Psi}^\dagger(x) \cdot \nabla\hat{\Psi}(\mathbf{x}) \\ &+ \frac{n}{2} \iint d^3\mathbf{x}\, d^3\mathbf{y}\, V_2(\mathbf{x}-\mathbf{y}) \times \left(\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y}) + \hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}^\dagger(\mathbf{y}) + 2\hat{\Psi}^\dagger(\mathbf{x})\hat{\Psi}(\mathbf{y}) \right).\end{aligned} \tag{9}$$

(b) As a Hamiltonian, $\hat{H}_2$ describes free quasiparticles. To make this manifest, rewrite (9) in terms of the $\mathbf{k}$–modes of the shifted fields, then perform a Bogolyubov transform to bring it to the form (6) with frequencies

$$\omega_{\mathbf{k}} = |\mathbf{k}| \times \sqrt{\frac{n}{M} W(\mathbf{k}) + \frac{\mathbf{k}^2}{4M^2}} \qquad \text{where} \quad W(\mathbf{k}) = \int d^3\mathbf{x}\, e^{i\mathbf{k}\mathbf{x}} V_2(\mathbf{x}). \tag{10}$$

In helium, $W(\mathbf{k})$ decreases with momentum $\mathbf{k}$ so rapidly that the dispersion relation (10) has a dip,



(11)

For small momenta, $\omega \approx c_s|\mathbf{k}|$ and the quasiparticles are *phonons.* For large momenta, $\omega \approx \mathbf{k}^2/2M$ and the quasiparticles are *helium atoms knocked out of Bose condensate.* And for intermediate momenta, the quasiparticles interpolate between the phonons and the atoms. In the region where the curve $\omega(k)$ dips down, the quasiparticles are called *rotons*; the historical reasons for this name turned out to be wrong, but the name stuck.

To see how this works, note that the *quasiparticles* are created by the $\hat{b}^\dagger_{\mathbf{k}}$ operators and annihilated by the $\hat{b}_{\mathbf{k}}$. Thus, the quasiparticle vacuum is not the no-atoms state $|0\rangle$ but the ground state of the Hamiltonian (6), which is the unique state $|\Omega\rangle$ annihilated by all the $\hat{b}_{\mathbf{k}}$ operators, $\hat{b}_{\mathbf{k}}\,|\Omega\rangle = 0$. The excited states have some quasiparticles on top of this $|\Omega\rangle$, *i.e.* $\big|N_{\text{QP}} : \mathbf{k}_1, \ldots, \mathbf{k}_n\big\rangle \propto \hat{b}^\dagger_{\mathbf{k}_n} \cdots \hat{b}^\dagger_{\mathbf{k}_1}\,|\Omega\rangle$. Note that according to eq. (5), the quasiparticles have definite mechanical momenta. On the other hand, they do not have well-defined atomic numbers because the phase symmetry generated by the $\hat{N}$ is spontaneously broken.

(c) Check that for large momenta $\hat{b}^\dagger_{\mathbf{k}} \approx \hat{a}^\dagger_{\mathbf{k}}$ and therefore the quasi-particle is approximately an atom, while for small momenta $\hat{b}^\dagger_{\mathbf{k}} \approx (\text{coeff}) \times (\hat{a}^\dagger_{\mathbf{k}} + \hat{a}_{\mathbf{k}})$ and therefore the quasiparticle is approximately a phonon.

Finally, let us consider a moving superfluid. For simplicity, we assume uniform motion with velocity $\mathbf{v}$. As discussed in class, this motion is described by

$$\Phi(\mathbf{x}) \equiv \langle\hat{\Psi}(\mathbf{x})\rangle = \sqrt{n} \times \exp(iM\mathbf{v}\cdot\mathbf{x}), \tag{12}$$

so let's define the shifted quantum fields according to

$$\hat{\Psi}(\mathbf{x}) = e^{iM\mathbf{vx}} \times \left(\sqrt{n} + \delta\hat{\Psi}(\mathbf{x})\right), \qquad \hat{\Psi}^\dagger(\mathbf{x}) = e^{-iM\mathbf{vx}} \times \left(\sqrt{n} + \delta\hat{\Psi}^\dagger(\mathbf{x})\right). \tag{13}$$

Physically, the $e^{\pm iM\mathbf{vx}}$ factors multiplying the shifted fields mean that the latter describe fluctuations in the frame of the moving superfluid rather than in the lab frame.

(d) Write the free-energy operator of the moving superfluid in terms of the shifted fields and show that

$$\hat{H} - \mu'\hat{N} = \text{const} + \hat{H}_2 + \mathbf{v}\cdot\hat{\mathbf{P}} + \hat{H}_{3+4} \tag{14}$$

where $\hat{H}_2$ and $\hat{H}_{3+4}$ are exactly as for the superfluid at rest, and the momentum operator is exactly as in eq. (5). Note that the chemical potential here includes the kinetic energy of the uniform motion, $\mu' = \mu + \frac{1}{2}M\mathbf{v}^2$.

As far as the quasiparticles in a moving superfluid are concerned, the free part of their Hamiltonian comprises all the quadratic terms in the operator (14), thus

$$\hat{H}_2' = \hat{H}_2 + \mathbf{v}\cdot\hat{\mathbf{P}} = \sum_{\mathbf{k}} (\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k})\, \hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}\,. \tag{15}$$

which acts as a Hamiltonian of free quasiparticles As discussed in class, any mode $\mathbf{k}$ with a negative coefficient of the $\hat{b}_{\mathbf{k}}^\dagger \hat{b}_{\mathbf{k}}$ operator would have a spontaneous buildup of quasiparticles, which would in turn lead to dissipation of the fluid's motion and hence loss of superfluidity. However, for the dispersion relation (10), all the coefficients $(\omega_{\mathbf{k}} + \mathbf{v}\cdot\mathbf{k})$ are positive as long as the fluid flows slower than some critical speed

$$v_c = \min_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{|\mathbf{k}|} > 0, \tag{16}$$

For $|\mathbf{v}| < v_c$ there is no spontaneous buildup of quasiparticles and hence no dissipation of the superflow. *This is why the superfluid flows without resistance.*

3. The last exercise is about relativistic theories. When an *exact* symmetry of a quantum field theory is spontaneously broken down, it gives rise to exactly massless Goldstone bosons. But when the spontaneously broken symmetry was only approximate to begin with, the would-be Goldstone bosons are no longer exactly massless but only relatively light. The best-known examples of such pseudo-Goldstone bosons are the pi-mesons $\pi^\pm$ and π^0, which are indeed much lighter then other hadrons. The Quantum ChromoDynamics theory (QCD) of strong interactions has an approximate chiral isospin symmetry $SU(2)_L \times SU(2)_R \cong \mathrm{Spin}(4)$. This symmetry would be exact if the two lightest quark flavors u and d were massless; in real life, the masses m_u and m_d are small but non quite zero, and the symmetry is only approximate. Somehow (and people are still arguing how), the chiral isospin symmetry is spontaneously broken down to the ordinary isospin symmetry $SU(2) \cong \mathrm{Spin}(3)$, and the 3 generators of the broken $\mathrm{Spin}(4)/\mathrm{Spin}(3)$ give rise to 3 (pseudo) Goldstone bosons $\pi^\pm$ and π^0.

   QCD is a rather complicated theory, so it is often convenient to describe the physics of the spontaneously broken chiral symmetry in terms of a simpler theory with similar symmetries. For example, the *linear sigma model* is a theory of 4 real scalar fields, an isosinglet $\sigma(x)$ and an isotriplet $\underset{\sim}{\pi}(x)$ comprising $\pi^1(x)$, $\pi^2(x)$ and $\pi^3(x)$ (or equivalently, $\pi^0(x) \equiv \pi^3(x)$ and $\pi^\pm(x) \equiv \big(\pi^1(x) \pm i\pi^2(x)\big)/\sqrt{2}$). The Lagrangian

$$\mathcal{L} \;=\; \tfrac{1}{2}(\partial_\mu \sigma)^2 \;+\; \tfrac{1}{2}(\partial_\mu \underset{\sim}{\pi})^2 \;-\; \frac{\lambda}{8}\left(\sigma^2 + \underset{\sim}{\pi}^2 - f^2\right)^2 \;+\; \beta\lambda f^2 \times \sigma \tag{17}$$

   is invariant under the $SO(4)$ rotations of the four fields, except for the last term which we treat as a perturbation. In class we saw that for $\beta = 0$ this theory has $SO(4)$ spontaneously broken to $SO(3)$ and hence 3 massless Goldstone bosons — the pions. In this exercise, we let $\beta > 0$ but $\beta \ll f$ to show how this leads to pions being massive but light.

   (a) Show that the scalar potential of the linear sigma model with $\beta > 0$ has a unique minimum at

$$\langle \underset{\sim}{\pi} \rangle \;=\; 0 \quad \text{and} \quad \langle \sigma \rangle \;=\; f \;+\; \beta \;+\; O(\beta^2/f). \tag{18}$$

   (b) Expand the fields around this minimum and show that the pions are light while the σ particle is much heavier. Specifically, $M_\pi^2 \approx \lambda f \beta$ while $M_\sigma^2 \approx \lambda f(f+\beta) \approx \lambda f^2 \gg M_\pi^2$.