Note: Continuous symmetries of any field theory form two separate groups: (1) the spacetime symmetries (Lorentz and translations); (2) the internal symmetries, which mix the fields with each other but do not move them in spacetime. In this homework we focus on the internal symmetries only.

1. Consider a theory of N complex scalar fields $\phi_a(x)$, $a = 1, 2, \ldots, N$ and a vector field $A_\mu(x)$. The Lagrangian is

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \;+\; \sum_a D^\mu\phi_a^*\,D_\mu\phi_a \;-\; \frac{\lambda}{4}\left(\sum_a |\phi_a|^2\right)^2 \;-\; m^2\sum_a |\phi_a|^2 \tag{1}$$

where $D_\mu\phi_a(x) = \partial_\mu\phi_a(x) + ieA_\mu(x)\phi_a(x)$; note all the ϕ_a have the same charge e.

(a) Show that the internal symmetry group of this theory is $G = U(1)^{\text{local}} \times SU(N)^{\text{global}}$.

From now on, $m^2 < 0$ and the scalar potential of the theory is minimized for $\phi \neq 0$. Thanks to the $SU(N)$ symmetry, such minima form a sphere in $\mathbf{C}^N = \mathbf{R}^{2N}$, and without loss of generality we shall focus on the "North pole" of this sphere, *i.e.* a point where $\phi_i = 0$ for $i = 1, \ldots, (N-1)$ while ϕ_N is real and positive.

(b) Show that symmetries preserving this minimum form subgroup $[SU(N-1)\times U(1)]^{\text{global}}$ of G.

(c) Use Nambu–Goldstone theorem to determine how many massless particles should the theory have and what are their quantum numbers with respect to the un-broken symmetries. Mind the Higgs mechanism for the local symmetries.

(d) And now let's derive the actual particle spectrum of the theory. Go to the unitary gauge for the local $U(1)$ symmetry, shift all fields by their vacuum expectation values, and expand the Lagrangian (1) in terms of the shifted fields. Use the quadratic part of this expansion to identify particle species and their masses.

(e) Focus on the the massless particles obtained in part (d) and make sure their quantum numbers agree with part (c).

2. Here is a more complicated problem on spontaneous symmetry breaking. Consider an $N \times N$ matrix $\Phi(x)$ of complex scalar fields ${\Phi^i}_j(x)$, $i, j = 1, \ldots, N$. In matrix notations, the Lagrangian is

$$\mathcal{L} \;=\; \mathrm{tr}\left(\partial^\mu \Phi^\dagger \,\partial_\mu \Phi\right) \;-\; V(\Phi^\dagger \Phi) \tag{2}$$

where the potential is

$$V \;=\; \frac{\alpha}{2}\,\mathrm{tr}\left(\Phi^\dagger \Phi \Phi^\dagger \Phi\right) \;+\; \frac{\beta}{2}\left(\mathrm{tr}\left(\Phi^\dagger \Phi\right)\right)^2 \;+\; m^2\,\mathrm{tr}\left(\Phi^\dagger \Phi\right). \tag{3}$$

(a) Show that this theory has global symmetry group $G = SU(N)_L \times SU(N)_R \times U(1)$ acting as

$$\Phi(x) \;\to\; e^{i\theta} U_L \Phi(x) U_R^\dagger, \qquad U_L, U_R \in SU(N). \tag{4}$$

($\star$) *Optional exercise, only for experts in group theory:*
Show that the theory has no other continuous symmetries besides G and Poincare (Lorentz and translations of spacetime).

From now on, we take $\alpha, \beta > 0$ but $m^2 < 0$. In this regime, V is minimized for non-zero vacuum expectation values $\langle \Phi \rangle \neq 0$ of the scalar fields.

(b) Let $(\kappa_1, \ldots, \kappa_N)$ be eigenvalues of the hermitian matrix $\Phi^\dagger \Phi$. Express the potential (3) in terms of these eigenvalues and show that the minimum lies at

$$\kappa_1 \;=\; \kappa_2 \;=\; \cdots \;=\; \kappa_N \;=\; C^2 \;=\; \frac{-m^2}{\alpha + N\beta} \;>\; 0. \tag{5}$$

In terms of the matrix Φ, eq. (5) means $\Phi = C \times$ a unitary matrix. All such minima are related by symmetries (4) to $\Psi = C \times$ the unit matrix, so without loss of generality we may assume that the vacuum lies at

$$\langle \Phi \rangle \;=\; C \times \mathbf{1}_{N\times N} \quad i.e. \quad \left\langle {\Phi^i}_j \right\rangle \;=\; C \times \delta^i_j\,. \tag{6}$$

(c) Show that in this vacuum, the symmetry group of the theory is spontaneously broken down to $SU(N)$; in terms of eq. (4), the unbroken symmetries have $U_L = U_R \in SU(N)$ and $\theta = 0$.

Let's expand the theory around the vacuum (6). For convenience, let's also decompose the complex matrix Φ into its hermitian and anti-hermitian parts,

$$\Phi(x) \;=\; C\times \mathbf{1}_{N\times N} \;+\; \frac{\varphi_1(x)\,+\,i\,\varphi_2(x)}{\sqrt{2}} \qquad \text{where } \varphi_1^\dagger\equiv\varphi_1 \text{ and } \varphi_2^\dagger\equiv\varphi_2\,. \tag{7}$$

(d) Expand the Lagrangian in powers of φ_1 and φ_2 and use the quadratic part $\mathcal{L}_2$ to determine the particle spectrum of the theory.

(e) Check the quantum numbers of the massless particles and verify that they agree with the Nambu–Goldstone theorem for the spontaneously broken symmetries of the theory.

3. Now let's gauge the $SU(N)_L\times SU(N)_R\times U(1)$ symmetry of the previous problem. Naturally, this requires gauge fields $B_\mu(x)$ and matrix-valued $L_\mu(x)$ and $R_\mu(x)$. In components, $L_\mu(x)=\sum_a \frac12\lambda^a\times L^a_\mu(x)$ and $R_\mu(x)=\sum_a \frac12\lambda^a\times R^a_\mu(x)$ where $a=1,\ldots,(N^2-1)$ and λ^a are the Gell–Mann matrices of $SU(N)$. The Lagrangian now is

$$\mathcal{L} \;=\; -\tfrac14 B_{\mu\nu}B^{\mu\nu} \;-\; \tfrac12\,\mathrm{tr}\,(L_{\mu\nu}L^{\mu\nu}) \;-\; \tfrac12\,\mathrm{tr}\,(R_{\mu\nu}R^{\mu\nu}) \;+\; \mathrm{tr}\left(D^\mu\Phi^\dagger\,D_\mu\Phi\right) \;-\; V(\Phi^\dagger\Phi) \tag{8}$$

where the scalar potential V is as in eq. (3), and

$$\begin{aligned}
B_{\mu\nu} &= \partial_\mu B_\nu \;-\; \partial_\nu B_\mu\,,\\
L_{\mu\nu} &= \partial_\mu L_\nu \;-\; \partial_\nu L_\mu \;+\; ig[L_\mu,L_\nu],\\
R_{\mu\nu} &= \partial_\mu R_\nu \;-\; \partial_\nu R_\mu \;+\; ig[R_\mu,R_\nu],\\
D_\mu\Phi &= \partial_\mu\Phi \;+\; ig'B_\mu\Phi \;+\; igL_\mu\Phi \;-\; ig\Phi R_\mu\,,\\
D_\mu\Phi^\dagger &= (D_\mu\Phi)^\dagger \;=\; \partial_\mu\Phi^\dagger \;-\; ig'B_\mu\Phi^\dagger \;+\; igR_\mu\Phi^\dagger \;-\; ig\Phi^\dagger L_\mu\,.
\end{aligned} \tag{9}$$

For simplicity, I assume equal gauge couplings $g_L=g_R=g$ for the two $SU(N)$ factors of the gauge group, but the abelian coupling g' is different.

As in the previous problem, we take $\alpha,\beta>0$ but $m^2<0$ so the scalar's vacuum expectation values $\langle\Phi\rangle$ are as in eq. (6), and the $SU(N)_L\times SU(N)_R\times U(1)$ gauge symmetry is broken down to $SU(N)$.

(a) Write down the mass matrix for the vector fields. Show that B_μ and $X^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu - R^a_\mu)$ vectors become massive while $V^a_\mu = \frac{1}{\sqrt{2}}(L^a_\mu + R^a_\mu)$ remain massless.

(b) Find the effective Lagrangian for the massless vector fields $V^a_\mu(x)$ by freezing all the other fields, *i.e.* setting $\Phi(x) \equiv \langle\Phi\rangle$, $B_\mu(x) \equiv 0$ and $X^a_\mu(x) \equiv 0$. Show that this Lagrangian describes a Yang–Mills theory with gauge group $SU(N)_V$ and gauge coupling $g_V = g/\sqrt{2}$.

($\star$) *Optional exercise:*
Rewrite the Lagrangian (8) in terms of fields of definite mass — V_μ, X_μ, B_μ and $\delta\Psi$ — and their derivatives that are covariant with respect to the unbroken $SU(N)_V$. For simplicity, fix the unitary gauge for the broken symmetries by demanding $\Phi^\dagger \equiv \Psi$, or in terms of eq. (7), freeze $\varphi_2 \equiv 0$.

4. Finally, a problem on a different subject. Quantum mechanics of a fixed number of *relativistic* particles does not work (except as an approximation) because of problems with relativistic causality. Indeed, consider a single free relativistic spinless particle with Hamiltonian

$$\hat{H} \;=\; +\sqrt{M^2 + \hat{\mathbf{P}}^2} \tag{10}$$

(in the $c = \hbar = 1$ units). In the coordinate picture, this Hamiltonian is a horrible integro-differential operator, but that's only a technical problem. The real problem concerns the time evolution kernel

$$U(\mathbf{x}-\mathbf{y};t) \;=\; \langle \mathbf{x},t|\mathbf{y},t_0=0\rangle^{\text{Heisenberg}}_{\text{picture}} \;=\; \langle \mathbf{x}|\exp(-it\hat{H})\,|\mathbf{y}\rangle^{\text{Schroedinger}}_{\text{picture}}\,. \tag{11}$$

(a) Show that

$$U(\mathbf{x}-\mathbf{y};t) \;=\; \frac{-i}{4\pi^2\, r}\int\! dk\, k\, \exp\bigl(irk - it\omega(k)\bigr), \tag{12}$$

where $r = |\mathbf{x}-\mathbf{y}|$ and $\omega(k) = \sqrt{M^2+k^2}$.

(b) Take the limit $t \to \infty$, $r \to \infty$, with fixed ratio r/t; let's stay inside the future light cone, so $(r/t) < 1$. Show that in this limit, the evolution kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \;\approx\; \frac{(-iM)^{3/2}}{4\pi^{3/2}}\,\frac{t}{(t^2-r^2)^{5/4}} \times \exp(-iM\sqrt{t^2-r^2}). \tag{13}$$

Hint: Use the saddle point method to evaluate the integral (12). If you are not familiar with this method, see the `mathematical supplement.`

(c) Finally, take a similar limit but go outside the light cone, thus fixed $(r/t) > 1$ while $R, t \to \infty$. Show that in this limit, the kernel becomes

$$U(\mathbf{x}-\mathbf{y};t) \;\approx\; \frac{iM^{3/2}}{4\pi^{3/2}}\,\frac{t}{(r^2-t^2)^{5/4}} \times \exp(-M\sqrt{r^2-t^2}). \tag{14}$$

This formula shows that the kernel diminishes exponentially outside the light cone, *but it does not vanish!* Thus, given a particle localized at point $\mathbf{y}$ at the time $t_0 = 0$, after time $t > 0$, its wave function is *mostly* limited to the future light cone $r < t$, *but there is an exponential tail outside the light cone.* In other words, the probability of superluminal motion is exponentially small but non-zero.

Obviously, such superluminal propagation cannot be allowed in a consistently relativistic theory. And that's why relativistic quantum mechanics of a single particle is inconsistent. Likewise, relativistic quantum mechanics of any fixed number of particles does not work, except as an approximation.

In the quantum field theory, this paradox is resolved by allowing for creation and annihilation of particles. Quantum field operators acting at points x and y outside each others' lightcones can either create a particle at x and then annihilate it at y, or else annihilate it at y and then create it at x. I will show in class that the two effects *precisely* cancel each other, so altogether there is no propagation outside the light cone. That's how relativistic QFT is perfectly causal while the relativistic QM is not.