1. First, finish the previous homework set and do problem #4.

   To evaluate some integrals in that problem, you need the *saddle point method*. If you are not familiar with this method — or any of the related methods for approximating integrals of the form $\int dx\, f(x) \times \exp(Ag(x))$ for $A \to \infty$ — then *read my mathematical supplement on the subject.*

2. The second problem is about quantum massive vector field $A_\mu(x)$ and its expansion into creation and annihilation operators. The classical massive vector fields has appeared in two earlier homeworks sets: in set #1 we derived the equations of motions from the Lagrangian, and in set #3 we developed the Hamiltonian formalism. The canonical quantization of that formalism is completely straightforward: The 3–vector field $\mathbf{A}(x)$ and its canonical conjugate $-\mathbf{E}(x)$ become quantum fields obeying equal-time commutation relations

$$[\hat{A}^i(\mathbf{x}), \hat{A}^j(\mathbf{y})] \;=\; 0, \quad [\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \;=\; 0, \quad [\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})] \;=\; -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{1}$$

(in $\hbar = 1, c = 1$ units), the time-independent equation of motion for the A_0 field becomes an operatorial identity

$$\hat{A}^0(x) \;=\; -\frac{\nabla\cdot\hat{\mathbf{E}}(x)}{m^2}\,, \tag{2}$$

and the Hamiltonian operator

$$\hat{H} \;=\; \int\! d^3\mathbf{x}\left(\tfrac{1}{2}\hat{\mathbf{E}}^2 \;+\; \frac{(\nabla\cdot\hat{\mathbf{E}})^2}{2m^2} \;+\; \tfrac{1}{2}(\nabla\times\hat{\mathbf{A}})^2 \;+\; \tfrac{1}{2}m^2\hat{\mathbf{A}}^2\right). \tag{3}$$

follows from the classical Hamiltonian. In this exercise, we assume free fields not coupled to any currents; otherwise, there would be additional terms involving $J^\mu(x)$ in eqs. (2) and (3).

In general, a QFT has a creation $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and an annihilation operator $\hat{a}_{\mathbf{k},\lambda}$ for each plane wave with momentum $\mathbf{k}$ and polarization λ. The massive vector fields have 3 independent polarizations corresponding to orthogonal 3–vectors. One may use any basis of 3 such vectors $\mathbf{e}_\lambda(\mathbf{k})$, and it's often convenient to make them $\mathbf{k}$–dependent and complex; in the complex case, orthogonality and the unit length mean

$$\mathbf{e}_\lambda(\mathbf{k}) \cdot \mathbf{e}^*_{\lambda'}(\mathbf{k}) \;=\; \delta_{\lambda,\lambda'} \,. \tag{4}$$

Of particular convenience is the helicity basis of eigenvectors of the vector product $i\mathbf{k}\times$, namely

$$i\mathbf{k}\times\mathbf{e}_\lambda(\mathbf{k}) \;=\; \lambda|\mathbf{k}|\mathbf{e}_\lambda(\mathbf{k}), \qquad \lambda = -1, 0, +1. \tag{5}$$

By convention, the phases of the complex helicity eigenvectors are chosen such that

$$\mathbf{e}_0(\mathbf{k}) \;=\; \frac{\mathbf{k}}{|\mathbf{k}|}\,, \quad \mathbf{e}^*_\lambda(\mathbf{k}) \;=\; (-1)^\lambda \mathbf{e}_{-\lambda}(\mathbf{k}), \quad \mathbf{e}_\lambda(-\mathbf{k}) \;=\; -\mathbf{e}^*_\lambda(+\mathbf{k}). \tag{6}$$

As a first step towards constructing the $\hat{a}_{\mathbf{k},\lambda}$ and $\hat{a}^\dagger_{\mathbf{k},\lambda}$ operators, we Fourier transform the vector fields $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ and then decompose the vectors $\hat{\mathbf{A}}_\mathbf{k}$ and $\hat{\mathbf{E}}_\mathbf{k}$ into helicity components,

$$\begin{aligned}
\hat{\mathbf{A}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda}\,, \qquad \hat{A}_{\mathbf{k},\lambda} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{A}}(\mathbf{x}),\\
\hat{\mathbf{E}}(\mathbf{x}) \;&=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda e^{i\mathbf{k}\mathbf{x}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{E}_{\mathbf{k},\lambda}\,, \qquad \hat{E}_{\mathbf{k},\lambda} \;=\; \int\! d^3\mathbf{x}\, e^{-i\mathbf{k}\mathbf{x}}\mathbf{e}^*_\lambda(\mathbf{k})\cdot\hat{\mathbf{E}}(\mathbf{x}).
\end{aligned} \tag{7}$$

(a) Show that $\hat{A}^\dagger_{\mathbf{k},\lambda} = -\hat{A}_{-\mathbf{k},\lambda}$, $\hat{E}^\dagger_{\mathbf{k},\lambda} = -\hat{E}_{-\mathbf{k},\lambda}$, and derive the equal-time commutation relations for the $\hat{A}_{\mathbf{k},\lambda}$ and $\hat{E}_{\mathbf{k},\lambda}$ operators.

(b) Show that

$$\hat{H} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda\left(\frac{C_{\mathbf{k},\lambda}}{2}\,\hat{E}^\dagger_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda} \;+\; \frac{\omega^2_\mathbf{k}}{2C_{\mathbf{k},\lambda}}\hat{A}^\dagger_{\mathbf{k},\lambda}\hat{A}_{\mathbf{k},\lambda}\right) \tag{8}$$

where $\omega_\mathbf{k} = \sqrt{\mathbf{k}^2+m^2}$ and $C_{\mathbf{k},\lambda} = 1+\delta_{\lambda,0}(\mathbf{k}^2/m^2)$.

(c) Define creation and annihilation operators according to

$$\hat{a}_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}\hat{A}_{\mathbf{k},\lambda} \;-\; iC_{\mathbf{k},\lambda}\hat{E}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}\,, \qquad \hat{a}^{\dagger}_{\mathbf{k},\lambda} \;=\; \frac{\omega_{\mathbf{k}}\hat{A}^{\dagger}_{\mathbf{k},\lambda} \;+\; iC_{\mathbf{k},\lambda}\hat{E}^{\dagger}_{\mathbf{k},\lambda}}{\sqrt{C_{\mathbf{k},\lambda}}}\,, \tag{9}$$

and verify that they satisfy equal-time bosonic commutation relations (relativistically normalized).

(d) Show that

$$\hat{H} \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_{\lambda}\omega_{\mathbf{k}}\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \text{const.} \tag{10}$$

(e) Next, consider the time dependence of the free vector field. Show that

$$\hat{\mathbf{A}}(\mathbf{x},t) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3\,2\omega_{\mathbf{k}}}\sum_{\lambda}\sqrt{C_{\mathbf{k},\lambda}}\left(e^{-ikx}\mathbf{e}_{\lambda}(\mathbf{k})\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}\mathbf{e}^{*}_{\lambda}(\mathbf{k})\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}}. \tag{11}$$

(f) Write down a similar formula for the $\hat{A}^0(\mathbf{x},t)$ (use eq. (2)). Together with the previous result, you should get

$$\hat{A}_{\mu}(x) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}\sum_{\lambda}\left(e^{-ikx}f_{\mu}(\mathbf{k},\lambda)\,\hat{a}_{\mathbf{k},\lambda}(0) \;+\; e^{+ikx}f^{*}_{\mu}(\mathbf{k},\lambda)\,\hat{a}^{\dagger}_{\mathbf{k},\lambda}(0)\right)_{k^0=+\omega_{\mathbf{k}}} \tag{12}$$

where the 4–vectors $f^{\mu}(\mathbf{k},\lambda)$ obtain by Lorentz boosting of purely-spatial polarization vectors $\mathbf{e}_{\lambda}(\mathbf{k})$ into the moving particle's frame. Specifically,

$$f^{\mu}(\mathbf{k},\lambda) \;=\; \begin{cases} \big(0,\mathbf{e}_{\lambda}(\mathbf{k})\big) & \text{for } \lambda=\pm 1, \\[1ex] \left(\frac{|\mathbf{k}|}{m},\frac{\omega_{\mathbf{k}}}{m}\frac{\mathbf{k}}{|\mathbf{k}|}\right) & \text{for } \lambda=0, \end{cases} \tag{13}$$

and they satisfy

$$k_{\mu}f^{\mu}_{\mathbf{k},\lambda} \;=\; 0, \qquad f^{\mu}_{\mathbf{k},\lambda}\left(f^{*}_{\mathbf{k},\lambda'}\right)_{\mu} \;=\; -\delta_{\lambda,\lambda'}\,. \tag{14}$$

(g) Finally, verify that the quantum vector field (12) satisfies the free equations of motion $\partial_{\mu}\hat{A}^{\mu}(x)=0$ and $(\partial^2+m^2)\hat{A}^{\mu}(x)=0$; moreover, each mode in the expansion (12) satisfies the equations of motions without any help from the other modes.

3. The last problem concerns the Feynman propagator for the massive vector field.

(a) First, a lemma: Show that

$$\sum_{\lambda} f^{\mu}(\mathbf{k},\lambda) f^{\nu *}(\mathbf{k},\lambda) \;=\; -g^{\mu\nu} \;+\; \frac{k^{\mu}k^{\nu}}{m^2}\,. \tag{15}$$

(b) Next, calculate the "vacuum sandwich" of two vector fields and show that

$$\begin{aligned} \langle 0|\, \hat{A}^{\mu}(x)\hat{A}^{\nu}(y)\,|0\rangle \;&=\; \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_{\mathbf{k}}}\left[\left(-g^{\mu\nu}+\frac{k^{\mu}k^{\nu}}{m^2}\right)e^{-ik(x-y)}\right]_{k^0=+\omega_{\mathbf{k}}} \\ &=\; \left(-g^{\mu\nu}-\frac{\partial^{\mu}\partial^{\nu}}{m^2}\right)D(x-y). \end{aligned} \tag{16}$$

(c) And now, the Feynman propagator: Show that

$$\begin{aligned} G_F^{\mu\nu} \;\equiv\; \langle 0|\,\mathbf{T}^{*}\hat{A}^{\mu}(x)\hat{A}^{\nu}(y)\,|0\rangle \;&=\; \left(-g^{\mu\nu}-\frac{\partial^{\mu}\partial^{\nu}}{m^2}\right)G_F^{\mathrm{scalar}}(x-y) \\ &=\; \int\frac{d^4\mathbf{k}}{(2\pi)^4}\left(-g^{\mu\nu}+\frac{k^{\mu}k^{\nu}}{m^2}\right)\frac{ie^{-ik(x-y)}}{k^2-m^2+i0} \end{aligned} \tag{17}$$

where

$$\mathbf{T}^{*}\hat{A}^{\mu}(x)\hat{A}^{\nu}(y) \;=\; \mathbf{T}\hat{A}^{\mu}(x)\hat{A}^{\nu}(y) \;+\; \frac{i}{m^2}\,\delta^{\mu 0}\delta^{\nu 0}\delta^{(4)}(x-y), \tag{18}$$

is the *modified* time-ordered product of the vector fields. The purpose of this modification[⋆] is to absorb the $\delta^{(4)}(x-y)$ stemming from the $\partial_0\partial_0 G_F(x-y)$.

(d) Finally, write the classical action for the free vector field as

$$S \;=\; \tfrac{1}{2}\int d^4x\, A_{\mu}(x)\,\mathcal{D}^{\mu\nu}A_{\nu}(x) \tag{19}$$

where $\mathcal{D}^{\mu\nu}$ is a differential operator and show that the Feynman propagator (17) is a Green's function of this operator,

$$\mathcal{D}_x^{\mu\nu}G^F_{\nu\lambda} \;=\; +i\delta^{\mu}_{\lambda}\delta^{(4)}(x-y). \tag{20}$$

⋆ See *Quantum Field Theory* by Claude Itzykson and Jean–Bernard Zuber.