1. The first problem concerns quantizing the massless EM field $A^\mu(x)$. In the previous homework set, we had massive vector fields satisfying equal-times commutation relations

$$[\hat{A}^i(\mathbf{x}),\hat{A}^j(\mathbf{y})] \;=\; 0, \quad [\hat{E}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})] \;=\; 0, \quad [\hat{A}^i(\mathbf{x}),\hat{E}^j(\mathbf{y})] \;=\; -i\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{1}$$

But the massless electric field satisfies the Gauss Law

$$\nabla\cdot\hat{\mathbf{E}}(\mathbf{x}) \;=\; \rho(\mathbf{x}) \;\to\; 0 \text{ for free fields.} \tag{2}$$

This equation is time-independent, so in the quantum theory it becomes an operatorial identity. Consequently, $\nabla\cdot\hat{\mathbf{E}}(\mathbf{x})$ must commute with all the EM fields — including all the $\hat{A}^i(\mathbf{y})$ — and that contradicts the canonical commutation relations (1).

We need to resolve this contradiction, and at the same time make sure that gauge transforms do not affect the commutations relations. The simplest way to solve both problems is to *fix the Coulomb gauge* in which

$$\nabla\cdot\hat{\mathbf{A}}(\mathbf{x},t) \;\equiv\; 0. \tag{3}$$

This makes both $\hat{\mathbf{A}}$ and $\hat{\mathbf{E}}$ fields transverse, or in terms of momenta and helicities,

$$\hat{A}_{\mathbf{k},\lambda=0} \;\equiv\; \hat{E}_{\mathbf{k},\lambda=0} \;\equiv\; 0 \tag{4}$$

and hence

$$\hat{\mathbf{A}}(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm1}^{\textbf{only}} e^{i\mathbf{kx}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{A}_{\mathbf{k},\lambda}\,, \qquad \hat{\mathbf{E}}(\mathbf{x}) \;=\; \int\!\frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{\lambda=\pm1}^{\textbf{only}} e^{i\mathbf{kx}}\mathbf{e}_\lambda(\mathbf{k})\,\hat{E}_{\mathbf{k},\lambda}\,. \tag{5}$$

The transverse modes have canonical commutation relations

$$[\hat{A}_{\mathbf{k},\lambda},\hat{A}_{\mathbf{k}',\lambda'}] \,=\, 0, \qquad [\hat{E}_{\mathbf{k},\lambda},\hat{E}_{\mathbf{k}',\lambda'}] \,=\, 0, \qquad [\hat{A}_{\mathbf{k},\lambda},\hat{E}_{\mathbf{k}',\lambda'}] \,=\, i\delta_{\lambda,\lambda'}(2\pi)^3\delta^{(3)}(\mathbf{k}+\mathbf{k}') \tag{6}$$

for $\lambda,\lambda'=\pm1$ **only,** while the longitudinal modes are absent altogether. Finally, the $\hat{A}^0$ field satisfies $\nabla^2\hat{A}^0 = -\nabla\cdot\hat{\mathbf{E}}$ and hence $\hat{A}^0 \equiv 0$ *for free EM fields.*[⋆]

(a) Given this setup, define the photonic creation and annihilation operators $\hat{a}^\dagger_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$, verify that they satisfy the bosonic commutations relations, and expand the EM Hamiltonian as

$$\hat{H} \;=\; \int\! d^3\mathbf{x}\left(\tfrac{1}{2}\hat{\mathbf{E}}^2 \,+\, \tfrac{1}{2}\hat{\mathbf{B}}^2\right) \;=\; \int \frac{d^3\mathbf{k}}{(2\pi^3)}\,\frac{1}{2|\mathbf{k}|}\sum_{\lambda=\pm1}^{\textbf{only}} |\mathbf{k}|\times\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} \;+\; \text{const.} \tag{7}$$

(b) Go to Heisenberg picture and express time-dependent $\hat{A}^\mu(x)$ fields in terms of the $\hat{a}^\dagger_{\mathbf{k},\lambda}(0)$ and $\hat{a}_{\mathbf{k},\lambda}(0)$.

(c) Calculate the "vacuum sandwich" of two vector fields and show that

$$\langle 0|\,\hat{A}^\mu(x)\,\hat{A}^\nu(y)\,|0\rangle \;=\; \int \frac{d^3\mathbf{k}}{(2\pi^3)}\,\frac{1}{2|\mathbf{k}|}\,C^{\mu\nu}(k)\,e^{-ik(x-y)}\Big|_{k^0=+|\mathbf{k}|} \tag{8}$$

where

$$C^{ij}(k) \;=\; \delta^{ij} \;-\; \frac{k^ik^j}{\mathbf{k}^2}\,,\quad C^{0i} \;=\; C^{i0} \;=\; C^{00} \;=\; 0. \tag{9}$$

(d) Lemma: show that $C^{\mu\nu}(k) = -g^{\mu\nu} + k^\mu q^\nu + q^\mu k^\nu$ for some $q^\nu(k)$.

(e) Finally, derive the Feynman propagator for the EM fields in the Coulomb gauge,

$$G^{\mu\nu}_F(x-y) \;\equiv\; \langle 0|\,\mathbf{T}\hat{A}\mu(x)\,\hat{A}^\nu(y)\,|0\rangle \;=\; \int \frac{d^4k}{(2\pi)^4}\,\frac{ie^{-ik(x-y)}}{k^2\,+\,i\epsilon}\times C^{\mu\nu}(k). \tag{10}$$

Note: In other gauges, the EM Feynman propagator is given by formulae which look just like eq. (10) but with a different $C^{\mu\nu}(k)$ tensor. Generally, $C^{\mu\nu}(k) = -g^{\mu\nu} + k^\mu q^\nu + q^\mu k^\nu$ for some $q^\mu(k)$, but the specific form of the $q^\mu(k)$ depends on the gauge.

⋆ For the EM fields coupled to electric charges and currents, the $\mathbf{A}\equiv 0$ gauge condition implies $\nabla^2A^0(\mathbf{x},t) = -\rho(\mathbf{x},t)$ and hence

$$A^0(\mathbf{x},t) \;=\; \int\! d^3\mathbf{y}\,\frac{\rho(\mathbf{y},t)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

is the instantaneous Coulomb field of the electric charge density $\rho(\mathbf{y},t)$. That's why this gauge is called *the Coulomb gauge.*

2. Next, an exercise in Dirac matrices γ^μ. Please do not assume any specific form of these 4×4 matrices, just use the anti-commutation relations

$$\gamma^\mu \gamma^\nu \;+\; \gamma^\nu \gamma^\mu \;=\; 2g^{\mu\nu}. \tag{11}$$

In class, we have defined the spin matrices

$$S^{\mu\nu} \;=\; -S^{\nu\mu} \;\stackrel{\text{def}}{=}\; \tfrac{i}{4}\,[\gamma^\mu, \gamma^\nu] \tag{12}$$

and showed that

$$\left[S^{\mu\nu}, \gamma^\lambda\right] \;=\; ig^{\nu\lambda}\gamma^\mu \;-\; ig^{\mu\lambda}\gamma^\nu. \tag{13}$$

(a) Show that the spin matrices $S^{\mu\nu}$ have commutation relations of the Lorentz generators,

$$\left[S^{\kappa\lambda}, S^{\mu\nu}\right] \;=\; ig^{\lambda\mu}S^{\kappa\nu} \;-\; ig^{\lambda\nu}S^{\kappa\mu} \;-\; ig^{\kappa\mu}S^{\lambda\nu} \;+\; ig^{\kappa\nu}S^{\lambda\mu}. \tag{14}$$

Continuous Lorentz transforms obtain from integrating infinite sequences of infinitesimal transforms $X'^\mu \;=\; X^\mu + \epsilon\Theta^\mu{}_\nu X^\nu$ where $\Theta_{\mu\nu} \;=\; -\Theta_{\nu\mu}$. Altogether, a finite continuous transform acts as $X'^\mu = L^\mu{}_\nu X^\nu$ where

$$L \;=\; \exp(\Theta), \quad i.e., \quad L^\mu{}_\nu = \delta^\mu_\nu \;+\; \Theta^\mu{}_\nu \;+\; \tfrac{1}{2}\Theta^\mu{}_\lambda\Theta^\lambda{}_\nu \;+\; \tfrac{1}{6}\Theta^\mu{}_\kappa\Theta^\kappa{}_\lambda\Theta^\lambda{}_\nu \;+\; \cdots. \tag{15}$$

(b) Let L be a Lorentz transform of the form (15), and let $M(L) = \exp\left(-\frac{i}{2}\theta_{\alpha\beta}S^{\alpha\beta}\right)$.
Show that $M^{-1}(L)\gamma^\mu M(L) = L^\mu{}_\nu\gamma^\nu$.

Next, a little more algebra:

(c) Calculate $\{\gamma^\rho, \gamma^\lambda\gamma^\mu\gamma^\nu\}$, $[\gamma^\rho, \gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu]$ and $[S^{\rho\sigma}, \gamma^\lambda\gamma^\mu\gamma^\nu]$.

(d) Show that $\gamma^\alpha\gamma_\alpha = 4$, $\gamma^\alpha\gamma^\nu\gamma_\alpha = -2\gamma^\nu$, $\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu}$ and $\gamma^\alpha\gamma^\lambda\gamma^\mu\gamma^\nu\gamma_\alpha = -2\gamma^\nu\gamma^\mu\gamma^\lambda$.
Hint: use $\gamma^\alpha\gamma^\nu = 2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha$ repeatedly.

A charged spinor field $\Psi(x)$ in an EM background $A^\mu(x)$ satisfies gauge-covariant version of the Dirac equation, namely $\left(i\gamma^\mu D_\mu + m\right)\Psi(x) = 0$ where $D_\mu = \partial_\mu + iqA_\mu(x)$ are the covariant derivatives.

(e) Show that the this equation implies $\left(m^2 + D^2 + qF_{\mu\nu}S^{\mu\nu}\right)\Psi(x) = 0$.

3. Now consider the $\gamma^5 \stackrel{\text{def}}{=} i\gamma^0\gamma^1\gamma^2\gamma^3$ matrix.

   (a) Show that γ^5 anticommutes with each of the γ^μ matrices — $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ — and commutes with all the spin matrices $S^{\mu\nu}$.

   (b) Show that γ^5 is hermitian and that $(\gamma^5)^2 = 1$.

   (c) Show that $\gamma^5 = (-i/24)\epsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu$ and $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma^5$.

   (d) Show that $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = -i\epsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma^5$.

   (e) Show that any 4×4 matrix Γ is a unique linear combination of the following 16 matrices: 1, γ^μ, $\gamma^{[\mu}\gamma^{\nu]}$, $\gamma^5\gamma^\mu$, and γ^5.

   Conventions: $\epsilon^{0123} = +1$, $\epsilon_{0123} = -1$, $\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$,
   $\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]} = \frac{1}{6}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\lambda\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\lambda)$,
   and ditto for the $\gamma^{[\kappa}\gamma^\lambda\gamma^\mu\gamma^{\nu]}$.

   Now consider Dirac matrices in spacetime dimensions $d \neq 4$. Such matrices always satisfy the Clifford algebra (11), but their sizes depend on d.

   Let $\Gamma = i^n\gamma^0\gamma^1\cdots\gamma^{d-1}$ be the generalization of the γ^5 to d dimensions; the pre-factor $i^n = \pm i$ or ± 1 is chosen such that $\Gamma = \Gamma^\dagger$ and $\Gamma^2 = +1$.

   (f) *For even d*, Γ anticommutes with all the γ^μ. Prove this, and use this fact to show that there are 2^d independent products of the γ^μ matrices, and consequently the matrices should be $2^{d/2} \times 2^{d/2}$.

   (g) *For odd d*, Γ commutes with all the Γ^μ — prove this. Consequently, one can set $\Gamma = +1$ or $\Gamma = -1$; the two choices lead to in-equivalent sets of the γ^μ.

   Classify the independent products of the γ^μ for odd d and show that their net number is 2^{d-1}; consequently, the matrices should be $2^{(d-1)/2} \times 2^{(d-1)/2}$.

4. Because all the $S^{\mu\nu}$ matrices commute with the γ^5, all the $M(L)$ matrices are block-diagonal in the eigenbasis of the γ^5. In the Weyl convention for the γ matrices,

$$M(L) \;=\; \begin{pmatrix} M_L(L) & 0 \\ 0 & M_R(L) \end{pmatrix} \tag{16}$$

where all blocks are 2×2. This makes the Dirac spinor a *reducible representations* of the continuous Lorentz group $SO^+(3,1)$.

(a) Write down the explicit $S^{\mu\nu}$ matrices in the Weyl convention.

(b) Show that for a pure rotation through angle φ around axis $\mathbf{n}$,

$$M_L \;=\; M_R \;=\; \exp(-\tfrac{i}{2}\varphi\,\mathbf{n}\cdot\boldsymbol{\sigma}). \tag{17}$$

(c) Show that for a pure boost of *rapidity* r in the direction $\mathbf{n}$, $M_L = \exp(-\frac{r}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma})$ but $M_R = \exp(+\frac{r}{2}\,\mathbf{n}\cdot\boldsymbol{\sigma})$.
The rapidity is related to the β and γ parameters of a Lorentz boost as $\beta = \tanh(r)$, $\gamma = \cosh(r)$. For two successive boosts in the same directions, the rapidities add up, $r_{1+2} = r_1 + r_2$. In terms of the β and γ parameters,

$$M_L \;=\; \sqrt{\gamma} \times \sqrt{1 \,-\, \beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,, \qquad M_R \;=\; \sqrt{\gamma} \times \sqrt{1 \,+\, \beta\,\mathbf{n}\cdot\boldsymbol{\sigma}}\,. \tag{18}$$

The continuous Lorentz group — or rather its double cover Spin(3, 1) — is isomorphic to $SL(2,\mathbf{C})$, the group of complex 2×2 matrices (unitary or not) with det = 1. The $M_L(L)$ matrices act on the fundamental **2** multiplet of this group, while the $M_R(L)$ matrices act on the conjugate $\bar{\mathbf{2}}$ multiplet.

(d) Show that for any continuous Lorentz transform L,

$$\det M_L(L) \;=\; \det M_R(L) \;=\; 1 \qquad \text{and} \qquad M_R \;=\; \sigma_2 M_L^* \sigma_2\,. \tag{19}$$

Hint: for any Pauli matrix σ_i, $\sigma_2\sigma_i^*\sigma_2 = -\sigma_i$.