1. The first problem is about plane-wave solutions $e^{-ipx}u(p,s)$ and $e^{+ipx}v(p,x)$ of the Dirac equation. The 4–component spinors $u(p,s)$ and $v(p,s)$ satisfy

$$(\not{p}-m)u(p,s) \;=\; 0, \quad (\not{p}+m)v(p,s) \;=\; 0, \quad u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'}\,. \tag{1}$$

Let's start by writing down explicit formulae for these spinors in the Weyl basis for the $\gamma^\mu$ matrices.

(a) Show that for $\mathbf{p}=0$,

$$u(\mathbf{p}=\mathbf{0},s) \;=\; \begin{pmatrix} \sqrt{m}\,\xi_s \\ \sqrt{m}\,\xi_s \end{pmatrix} \tag{2}$$

where $\xi_s$ is a two-component $SO(3)$ spinor encoding the electron's spin state. The $\xi_s$ are normalized to $\xi_s^\dagger\xi_{s'}=\delta_{s,s'}$.

(b) For other momenta, $u(p,s)=M(\text{boost})u(\mathbf{p}=0,s)$ for the appropriate Lorentz boost. Use explicit formulae for the $M(\text{boost})$ from the previous homework (problem 8.4) to show that

$$u(p,s) \;=\; \begin{pmatrix} \sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \\ \sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\xi_s \end{pmatrix}. \tag{3}$$

(c) In class I argued that in the Weyl basis, $v(p,s)=\gamma^2u^*(p,s)$. Show that

$$v(p,s) \;=\; \begin{pmatrix} +\sqrt{E-\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \\ -\sqrt{E+\mathbf{p}\cdot\boldsymbol{\sigma}}\,\eta_s \end{pmatrix} \tag{4}$$

where $\eta_s=\sigma_2\xi_s^*=\pm i\xi_{-s}$. Note that $\eta^\dagger\mathbf{S}\eta=-\xi^\dagger\mathbf{S}\xi$ so $\eta_s$ has opposite spin from $\xi_s$; in Dirac sea terms, this corresponds to holes having opposite spins (as well as $p^\mu$) from the missing negative-energy particles.

(d) Show that for ultra-relativistic electrons or positrons of definite helicity $\lambda=\pm\frac{1}{2}$ Dirac plane waves become *chiral*, *i.e.* when you split the Dirac spinor into left-handed and right-handed Weyl spinors, one of the Weyl spinors becomes large while the other becomes negligibly small. Specifically,

$$
\begin{aligned}
u(p,-\tfrac{1}{2}) \;&\approx\; \sqrt{2E}\begin{pmatrix}\xi_L\\ 0\end{pmatrix}, \qquad & u(p,+\tfrac{1}{2}) \;&\approx\; \sqrt{2E}\begin{pmatrix}0\\ \xi_R\end{pmatrix},\\
v(p,-\tfrac{1}{2}) \;&\approx\; -\sqrt{2E}\begin{pmatrix}0\\ \eta_L\end{pmatrix}, \qquad & v(p,+\tfrac{1}{2}) \;&\approx\; \sqrt{2E}\begin{pmatrix}\eta_R\\ 0\end{pmatrix}.
\end{aligned}
\tag{5}
$$

Note that for electrons left/right chirality is same as helicity, but for positrons chirality is opposite from helicity.

Now lets focus on the properties of the $u$ and $v$ that do not depend on the Weyl basis, but you can use this basis to verify them.

(e) Verify that

$$u^\dagger(p,s)u(p,s') \;=\; v^\dagger(p,s)v(p,s') \;=\; 2E\delta_{s,s'} \tag{6}$$

and show that

$$\bar{u}^(p,s)u(p,s') \;=\; +2m\delta_{s,s'}\,, \qquad \bar{v}^(p,s)v(p,s') \;=\; -2m\delta_{s,s'}\,. \tag{7}$$

(f) Show that

$$\sum_{s=1,2} u_\alpha(p,s)\bar{u}_\beta(p,s) \;=\; (\not{p}+m)_{\alpha\beta} \quad \text{and} \quad \sum_{s=1,2} v_\alpha(p,s)\bar{v}_\beta(p,s) \;=\; (\not{p}-m)_{\alpha\beta}\,. \tag{8}$$

Hint: since there are only two independent spin states, $\sum_s \xi_s\xi_s^\dagger = 1_{2\times 2}$.

(g) Prove the Gordon identity

$$\bar{u}(p',s')\gamma^\mu u(p.s) \;=\; \frac{(p'+p)^\mu}{2m}\,\bar{u}(p',s')u(p,s) \;+\; \frac{i(p'-p)_\nu}{m}\,\bar{u}(p',s')S^{\mu\nu}u(p,s). \tag{9}$$

Hint: First, use Dirac equations for the $u$ and the $\bar{u}'$ to show that $2m\bar{u}'\gamma^\mu u = \bar{u}'(\not{p}'\gamma^\mu + \gamma^\mu\,\not{p})u$.

(h) Generalize the Gordon identity to $\bar{u}'\gamma^\mu v$, $\bar{v}'\gamma^\mu u$ and $\bar{v}'\gamma^\mu v$.

2. Now consider bilinear products of a Dirac field $\Psi(x)$ and its conjugate $\overline{\Psi}(x)$. Generally, such products have form $\overline{\Psi}\Gamma\Psi$ where $\Gamma$ is one of 16 matrices discussed in the previous homework (problem 8.3); altogether, we have

$$S = \overline{\Psi}\Psi, \quad V^\mu = \overline{\Psi}\gamma^\mu\Psi, \quad T^{\mu\nu} = \overline{\Psi}i\gamma^{[\mu}\gamma^{\nu]}\Psi, \quad A^\mu = \overline{\Psi}\gamma^5\gamma^\mu\Psi, \quad \text{and} \quad P = \overline{\Psi}i\gamma^5\Psi. \tag{10}$$

(a) Show that all the bilinears (10) are Hermitian.
Hint: First, show that $\left(\overline{\Psi}\,\Gamma\,\Psi\right)^\dagger = \overline{\Psi}\,\overline{\Gamma}\,\Psi$.
Note: despite the Fermi statistics, $\left(\Psi^\dagger_\alpha\Psi_\beta\right)^\dagger = +\Psi^\dagger_\beta\Psi_\alpha$.

(b) Show that under *continuous* Lorentz symmetries, the $S$ and the $P$ transform as scalars, the $V^\mu$ and the $A^\mu$ as vectors, and the $T^{\mu\nu}$ as an antisymmetric tensor.

(c) Find the transformation rules of the bilinears (10) under parity and show that while $S$ is a true scalar and $V$ is a true (polar) vector, $P$ is a pseudoscalar and $A$ is an axial vector.

Next, consider charge-conjugation properties of the Dirac bilinears. To avoid the operator-ordering problems, take $\Psi(x)$ and $\Psi^\dagger(x)$ to be "classical" fermionic fields which *anticommute* with each other, $\Psi_\alpha\Psi^\dagger_\beta = -\Psi^\dagger_\beta\Psi_\alpha$.

(d) In the Weyl convention, $\mathcal{C} : \Psi(x) \mapsto \pm\gamma^2\Psi^*(x)$. Show that $\mathcal{C} : \overline{\Psi}\Gamma\Psi \mapsto \overline{\Psi}\Gamma^c\Psi$ where $\Gamma^c = \gamma^0\gamma^2\Gamma^\top\gamma^0\gamma^2$.

(e) Calculate $\Gamma^c$ for all 16 independent matrices $\Gamma$ and find out which Dirac bilinears are $\mathcal{C}$–even and which are $\mathcal{C}$–odd.

(f) Verify that the Dirac action is invariant under charge conjugation.

3. Finally, a reading assignment: Read my notes about the *Spin–Statistics Theorem* at `http://bolvan.ph.utexas.edu/~vadim/Classes/2008f.homeworks/spinstat.pdf`