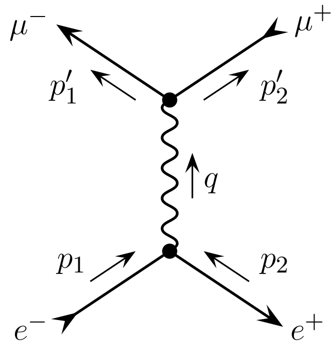
1. In class we discussed muon pair production in QED. At the tree level,



$$\left\langle \mu^-, \mu^+ \right| \mathcal{M} \left| e^-, e^+ \right\rangle \;=\; \frac{e^2}{s} \times \bar{u}(\mu^-)\gamma^\nu v(\mu^+) \times \bar{v}(e^+)\gamma_\nu u(e^-). \tag{1}$$

In class we have focused on un-polarized cross-section for this process; in this exercise we focus on amplitudes for definite helicities of all particles.

For simplicity, let us assume that all particles are ultra-relativistic so that the Dirac spinors $u(e^-)$, $v(e^+)$, $u(\mu^-)$, $v(\mu^+)$ all have definite chiralities,

$$\begin{aligned} u_L \;&\approx\; \sqrt{2E}\begin{pmatrix} \xi_L \\ 0 \end{pmatrix}, \qquad & u_R \;&\approx\; \sqrt{2E}\begin{pmatrix} 0 \\ \xi_R \end{pmatrix}, \\ v_L \;&\approx\; -\sqrt{2E}\begin{pmatrix} 0 \\ \eta_L \end{pmatrix}, \qquad & v_R \;&\approx\; \sqrt{2E}\begin{pmatrix} \eta_R \\ 0 \end{pmatrix}. \end{aligned} \tag{2}$$

cf. homework set#9, problem 1.

(a) Show that in the approximation (2)

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_L) \;=\; \bar{v}(e^+_R)\gamma_\nu u(e^-_R) \;=\; 0 \tag{3}$$

and hence we do not get any muon pairs produced unless the initial electron and positron have *opposite helicities*.

(b) Show that the μ^- and μ^+ must also have *opposite helicities* because

$$\bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_L) \;=\; \bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_R) \;=\; 0. \tag{4}$$

(c) Let's work in the center-of-mass frame where the initial e^- and e^+ collide along the z axis, $p_1^\nu = (E, 0, 0, +E)$, $p_2^\nu = (E, 0, 0, -E)$. Calculate the 4–vector $\bar{v}(e^+)\gamma^\nu u(e^-)$ in this frame and show that

$$\bar{v}(e^+_L)\gamma_\nu u(e^-_R) \;=\; 2E\times(0,+i,+1,0), \qquad \bar{v}(e^+_R)\gamma_\nu u(e^-_L) \;=\; 2E\times(0,-i,+1,0). \quad (5)$$

(d) In the CM frame the muons fly away on opposite directions at some angle θ to the electron / positron directions. Without loss of generality we may assume the muons momenta being in xz plane, thus

$$p_1'^\nu \;=\; (E,+E\sin\theta,0,+E\cos\theta), \qquad p_1'^\nu \;=\; (E,-E\sin\theta,0,-E\cos\theta) \qquad (6)$$

Calculate the 4–vector $\bar{u}(\mu^-)\gamma_\nu v(\mu^+)$ for the muons and show that

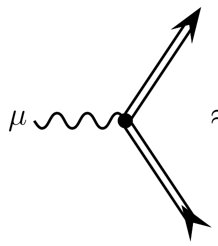
$$\begin{aligned} \bar{u}(\mu^-_R)\gamma^\nu v(\mu^+_L) &= 2E\times(0,-i\cos\theta,+1,+i\sin\theta), \\ \bar{u}(\mu^-_L)\gamma^\nu v(\mu^+_R) &= 2E\times(0,+i\cos\theta,+1,-i\sin\theta). \end{aligned} \qquad (7)$$

(e) Now calculate the amplitudes (1) for all possible combinations of particles' helicities, calculate the partial cross-sections, and show that

$$\begin{aligned} \frac{d\sigma(e^-_L+e^+_R\to\mu^-_L+\mu^+_R)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R+e^+_L\to\mu^-_R+\mu^+_L)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s}\times(1+\cos\theta)^2, \\ \frac{d\sigma(e^-_L+e^+_R\to\mu^-_R+\mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R+e^+_L\to\mu^-_L+\mu^+_R)}{d\Omega_{\text{c.m.}}} = \frac{\alpha^2}{4s}\times(1-\cos\theta)^2, \\ \frac{d\sigma(e^-_L+e^+_L\to\mu^-_{\text{any}}+\mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_R+e^+_R\to\mu^-_{\text{any}}+\mu^+_{\text{any}})}{d\Omega_{\text{c.m.}}} = 0, \\ \frac{d\sigma(e^-_{\text{any}}+e^+_{\text{any}}\to\mu^-_L+\mu^+_L)}{d\Omega_{\text{c.m.}}} &= \frac{d\sigma(e^-_{\text{any}}+e^+_{\text{any}}\to\mu^-_R+\mu^+_R)}{d\Omega_{\text{c.m.}}} = 0. \end{aligned} \qquad (8)$$

(f) Finally, sum / average over the helicities and calculate the un-polarized cross-section for muon pair production.

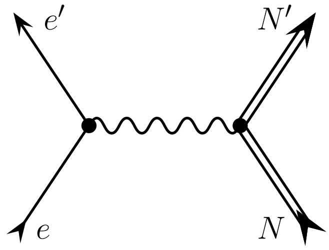
2. Now consider *Mott scattering* of a relativistic electron off a heavy nucleus of charge $+Ze$ and mass $M_N \gg m_e$. As long as the electron's energy E_e is no larger than a few tens of MeV, we may treat the nucleus as a point source of the electric field, and we may also neglect its recoil. Hence, in the CM frame — which is essentially the nucleus's frame — we may approximate the nucleus-nucleus-photon vertex as



$$\approx -iZe \times \begin{cases} 2M_N & \text{for } \mu = 0, \\ 0 & \text{for } \mu = 1,2,3. \end{cases} \tag{9}$$

To be precise, this formula includes the vertex and the external leg factors for the incoming and outgoing nucleus, but it does not include the photon's propagator.

In QED, there is only one tree diagram for the Mott scattering, namely



(10)

(a) Evaluate this diagram and write down the amplitude $\mathcal{M}$ in terms of $q = p' - p$ and $\bar{u}(p', s')\gamma^0 u(p, s)$.

(b) Assume the initial electron beam is un-polarized (*i.e.*, both values of spin s are equally likely) and the detector for the scattered electron does not measure its spin s' but only momentum s'. Show that for such an experiment,

$$\frac{d\sigma}{d\Omega} = \frac{(Z\alpha)^2}{(\mathbf{q}^2)^2} \times \frac{1}{2} \sum_{s,s'} \left| \bar{u}(p', s')\gamma^0 u(p, s) \right|^2 \tag{11}$$

where $\alpha = e^2/4\pi$ (or in conventional units, $\alpha = e^2/\hbar c$; anyhow, $\alpha \approx 1/137$.)

(c) Sum over the electron spins and show that

$$\frac{1}{2}\sum_{s,s'}\left|\bar{u}(p',s')\gamma^0 u(p,s)\right|^2 \;=\; 2\left(m_e^2 + EE' + \mathbf{p}\cdot\mathbf{p}'\right). \tag{12}$$

(d) Finally, assemble all the factors together and derive Mott formula

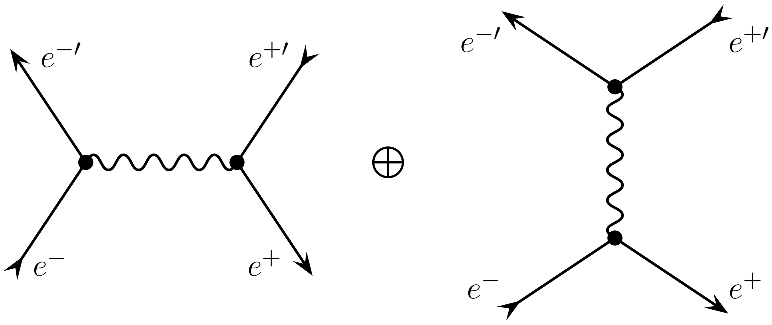
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \times \frac{1-\beta^2\sin^2(\theta/2)}{\gamma^2} \tag{13}$$

where β is the electron's speed (in $c = 1$ units), $\gamma = E/m_e$, and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{(Z\alpha)^2}{4m_e^2\beta^4\sin^4(\theta/2)} \tag{14}$$

is the classical Rutherford scattering cross-section (translated into $\hbar = c = 1$ units).

3. Finally consider the *Bhabha scattering* $e^-e^+ \to e^-e^+$. In QED, there are two tree-level Feynman diagrams contributing to this process, namely



(15)

(a) Evaluate the two diagrams and write the amplitude $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$. Mind the sign rules for the fermions.

Now comes the real work: calculating the un-polarized partial cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} = \frac{\overline{|\mathcal{M}|^2}}{64\pi^2 s} \tag{16}$$

where $\overline{|\mathcal{M}|^2}$ stands for $|\mathcal{M}|^2$ summed over final particle spins and averaged over the spins of the initial particles. Note the two diagrams (15) must be added together before squaring the amplitude, because

$$|\mathcal{M}_1+\mathcal{M}_2|^2 \;=\; |\mathcal{M}_1|^2 \;+\; |\mathcal{M}_2|^2 \;+\; 2\,\mathrm{Re}\big(\mathcal{M}_1^*\mathcal{M}_2\big) \;\neq\; |\mathcal{M}_1|^2 \;+\; |\mathcal{M}_2|^2. \qquad (17)$$

For simplicity, assume $E \gg m_e$ and neglect the electron's mass throughout your calculation. You may find it convenient to use Mandelstam's Lorentz-invariant kinematic variables s, t, and u, see eq. (5.69) of the Peskin & Schroeder textbook for details. In the $m_e \approx 0$ approximation, $p_1^2 = p_2^2 = p_1'^2 = p_2'^2 = m_e^2 \approx 0$ and hence

$$\begin{aligned}
s &\stackrel{\text{def}}{=} (p_1+p_2)^2 \equiv (p_1'+p_2')^2 \approx +2(p_1p_2) = +2(p_1'p_2'),\\
t &\stackrel{\text{def}}{=} (p_1'-p_1)^2 \equiv (p_2'-p_2)^2 \approx -2(p_1'p_1) = -2(p_2'p_2),\\
u &\stackrel{\text{def}}{=} (p_1'-p_2)^2 \equiv (p_2'-p_1')^2 \approx -2(p_1p_2') = -2(p_1'p_2),\\
&s+t+u \approx 0.
\end{aligned} \qquad (18)$$

(b) Sum / average over all spins the $|\mathcal{M}_2|^2$ of the second diagram and show that

$$\frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}_2|^2 \;=\; 2e^4 \times \frac{t^2+u^2}{s^2}\,. \qquad (19)$$

(c) Similarly, show that for the first diagram

$$\frac{1}{4}\sum_{\text{all spins}} |\mathcal{M}_1|^2 \;=\; 2e^4 \times \frac{s^2+u^2}{t^2}\,. \qquad (20)$$

(d) Now consider the interference $\mathcal{M}_1^* \times \mathcal{M}_2$ between the two diagrams. Show that

$$\frac{1}{4}\sum_{\text{all spins}} \mathcal{M}_1^* \times \mathcal{M}_2 \;=\; 2e^4 \times \frac{u^2}{st}\,. \qquad (21)$$

(e) Finally assemble all the terms together and show that for Bhabha scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{c.m.}} \;=\; \frac{\alpha^2}{2s} \times \frac{s^4+t^4+u^4}{s^2\times t^2} \;=\; \frac{\alpha^2}{4s} \times \left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2. \qquad (22)$$