1. First, a simple exercise about the Yukawa theory. For $M_s > 2m_f$ the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \to f + \bar{f}$.

   (a) Calculate the tree-level decay rate $\Gamma(S \to f + \bar{f})$.

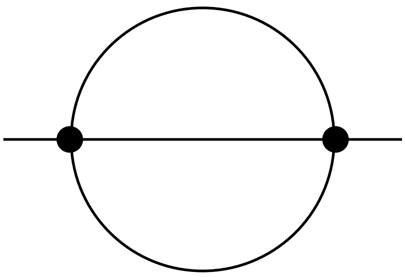
   (b) In class, we have calculated $\Sigma_\Phi^{1\,\mathrm{loop}}(p^2)$. Show that for $p^2 > 4m_f^2$ this function has an imaginary part and calculate it for $p^2 = M_s^2 + i\epsilon$.
   Note: at this level, you may neglect the difference between m_f^{bare} and m_f^{physical}.

   (c) Verify that

$$\mathrm{Im}\,\Sigma_\Phi^{1\,\mathrm{loop}}(p^2 = M_s^2 + i\epsilon) \;=\; -M_s\Gamma^{\mathrm{tree}}(S \to f + \bar{f}) \tag{1}$$

   and explain this relation in terms of the optical theorem.

The rest of this homework is about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory field strength renormalization begins at two-loop level. Specifically, the 1PI diagram



(2)

provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence to the $Z - 1$. *Your task is to evaluate this contribution. This is a difficult calculation, so proceed very carefully.*

2. First, use Feynman parameters to write the product of 3 propagators as

$$\prod_{j=1}^{3} \frac{i}{q_j^2 - m^2 + i0} \;=\; \iiint dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2i^3}{(\mathcal{D})^3} \tag{3}$$

   where

$$\mathcal{D} \;=\; xq_1^2 \;+\; yq_2^2 \;+\; zq_3^2 \;-\; m^2 \;+\; i0. \tag{4}$$

   Then impose $q_3 \equiv p - q_1 - q_2$ and shift the remaining 2 momentum variables from q_1 and

q_2 to $k_1 = q_1 + \cdots$ and $k_2 = q_2 + \cdots$ such that

$$\mathcal{D} \;=\; \alpha \times k_1^2 \;+\; \beta \times k_2^2 \;+\; \gamma \times p^2 \;-\; m^2 \;+\; i0 \tag{5}$$

for some (x, y, z)–dependent coefficients α, β, γ, for example

$$\alpha \;=\; (x+z), \qquad \beta \;=\; \frac{xy + xz + yz}{x+z}, \qquad \gamma \;=\; \frac{xyz}{xy + xz + yz}. \tag{6}$$

Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$.

Warning: Do not set $p^2 = m^2$ at this stage.

3. Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1\, d^4k_2\, \frac{1}{\mathcal{D}^4}. \tag{7}$$

Note that although this momentum integral diverges as $k_{1,2} \to \infty$, the divergence is logarithmic rather than quadratic.

4. To evaluate the momentum integral (7), first rotate both momenta k_1 and k_2 from Minkowski to Euclidean space, and then use dimensional regularization. You should get a formula looking like

$$\begin{aligned}\frac{d\Sigma}{dp^2} \;=\; \iiint & dxdydz\, \delta(x+y+z-1)\, F(x,y,z) \times \\ & \times \left\{ \frac{1}{\epsilon} \;+\; \log\frac{\mu^2}{m^2} \;+\; \text{const} \;+\; \log G(x,y,z;p^2/m^2) \right\}\end{aligned} \tag{8}$$

for some rational functions F and G of the Feynman parameters (and in case of G, also of p^2/m^2). Here are some useful formulæ for this problem:

$$\frac{6}{A^4} \;=\; \int_0^\infty dt\, t^3\, e^{-At}, \tag{9}$$

$$\int \frac{d^Dk}{(2\pi)^D}\, e^{-ctk^2} \;=\; \bigl(4\pi ct\bigr)^{-D/2}, \tag{10}$$

$$\Gamma(2\epsilon)X^\epsilon \;=\; \frac{1}{2\epsilon} \;-\; \gamma_E \;+\; \tfrac{1}{2}\log X \;+\; O(\epsilon). \tag{11}$$

5. Before you evaluate the Feynman parameter integral (8)— which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually

calculating the integrals

$$\iiint dxdydz\,\delta(x+y+z-1)\,F(x,y,z) \tag{12}$$

$$\text{and}\quad \iiint dxdydz\,\delta(x+y+z-1)\,F(x,y,z)\times\log G(x,y,z;p^2/m^2) \tag{13}$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $x, y \to 0$ while $z \to 1$ (or $x, z \to 0$, or $y, z \to 0$).

This calculation shows that

$$\frac{d\Sigma}{dp^2} \;=\; \frac{\text{constant}}{\epsilon} \;+\; \textit{a\_finite\_function}(p^2) \tag{14}$$

and hence

$$\begin{aligned}\Sigma(p^2) \;&=\; (\text{a divergent constant}) \;+\; (\text{another divergent constant})\times p^2 \\ &\qquad +\; \textit{a\_finite\_function}(p^2)\end{aligned} \tag{15}$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z \times p^2$.

6. Finally, let's use bare perturbation theory (bare λ and bare m^2 instead of the counterterms) and calculate field strength renormalization factor

$$Z \;=\; \left[1 \;-\; \frac{d\Sigma}{dp^2}\right]^{-1} \tag{16}$$

The derivative here should be evaluated at $p^2 = M^2_{\rm ph}$ — the physical mass^2 of the scalar particle, but to the leading approximation we may let $M^2_{\rm ph} \approx m^2$ and set $p^2 = m^2$ in eq. (8). This should simplify the $G(x,y,z)$ function, but the integral is still a big mess.

Do not try to evaluate the integrals (12) and (13) by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace the (x, y, z) variables with (w, ξ) according to

$$x = xi \times w, \quad y = (1 - \xi) \times w, \quad z = 1 - w,$$
$$\iiint_0^1 dxdydz\, \delta(x + y + z - 1) = \int_0^1 dw\, w \int_0^1 d\xi, \tag{17}$$

then integrate over the w variable first and over the ξ second. Here is a couple of integrals I did this way you might find useful:

$$\iiint dxdydz\, \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} = \frac{1}{2},$$
$$\iiint dxdydz\, \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} \times \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz)^2} = -\frac{3}{4}. \tag{18}$$

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $x + y + z = 1$, $x, y, z \geq 0$; modern computers can sum up to 10^8 grid points in just a few seconds. But watch out for singularities at the corners of the triangle.