

1. First, a simple exercise about the Yukawa theory. For $M_s > 2m_f$ the scalar particle becomes unstable: it decays into a fermion and an antifermion, $S \rightarrow f + \bar{f}$.

(a) Calculate the tree-level decay rate $\Gamma(S \rightarrow f + \bar{f})$.

(b) In class, we have calculated $\Sigma_{\Phi}^{1\text{loop}}(p^2)$. Show that for $p^2 > 4m_f^2$ this function has an imaginary part and calculate it for $p^2 = M_s^2 + i\epsilon$.

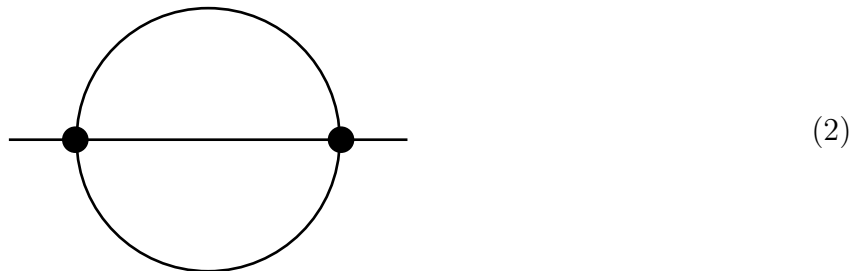
Note: at this level, you may neglect the difference between m_f^{bare} and m_f^{physical} .

(c) Verify that

$$\text{Im} \Sigma_{\Phi}^{1\text{loop}}(p^2 = M_s^2 + i\epsilon) = -M_s \Gamma^{\text{tree}}(S \rightarrow f + \bar{f}) \quad (1)$$

and explain this relation in terms of the optical theorem.

The rest of this homework is about the scalar $\lambda\phi^4$ theory. As discussed in class, in this theory field strength renormalization begins at two-loop level. Specifically, the 1PI diagram



provides the leading contribution to the $d\Sigma(p^2)/dp^2$ and hence to the $Z - 1$. *Your task is to evaluate this contribution. This is a difficult calculation, so proceed very carefully.*

2. First, use Feynman parameters to write the product of 3 propagators as

$$\prod_{j=1}^3 \frac{i}{q_j^2 - m^2 + i0} = \iiint dx dy dz \delta(x + y + z - 1) \frac{2i^3}{(\mathcal{D})^3} \quad (3)$$

where

$$\mathcal{D} = xq_1^2 + yq_2^2 + zq_3^2 - m^2 + i0. \quad (4)$$

Then impose $q_3 \equiv p - q_1 - q_2$ and shift the remaining 2 momentum variables from q_1 and

q_2 to $k_1 = q_1 + \dots$ and $k_2 = q_2 + \dots$ such that

$$\mathcal{D} = \alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m^2 + i0 \quad (5)$$

for some (x, y, z) -dependent coefficients α, β, γ , for example

$$\alpha = (x + z), \quad \beta = \frac{xy + xz + yz}{x + z}, \quad \gamma = \frac{xyz}{xy + xz + yz}. \quad (6)$$

Make sure the momentum shift has unit Jacobian $\partial(q_1, q_2)/\partial(k_1, k_2) = 1$.

Warning: Do not set $p^2 = m^2$ at this stage.

3. Express the derivative $d\Sigma(p^2)/dp^2$ in terms of

$$\iint d^4k_1 d^4k_2 \frac{1}{\mathcal{D}^4}. \quad (7)$$

Note that although this momentum integral diverges as $k_{1,2} \rightarrow \infty$, the divergence is logarithmic rather than quadratic.

4. To evaluate the momentum integral (7), first rotate both momenta k_1 and k_2 from Minkowski to Euclidean space, and then use dimensional regularization. You should get a formula looking like

$$\begin{aligned} \frac{d\Sigma}{dp^2} = & \iiint dx dy dz \delta(x + y + z - 1) F(x, y, z) \times \\ & \times \left\{ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} + \text{const} + \log G(x, y, z; p^2/m^2) \right\} \end{aligned} \quad (8)$$

for some rational functions F and G of the Feynman parameters (and in case of G , also of p^2/m^2). Here are some useful formulæ for this problem:

$$\frac{6}{A^4} = \int_0^\infty dt t^3 e^{-At}, \quad (9)$$

$$\int \frac{d^D k}{(2\pi)^D} e^{-ctk^2} = (4\pi ct)^{-D/2}, \quad (10)$$

$$\Gamma(2\epsilon) X^\epsilon = \frac{1}{2\epsilon} - \gamma_E + \frac{1}{2} \log X + O(\epsilon). \quad (11)$$

5. Before you evaluate the Feynman parameter integral (8)— which looks like a frightful mess — make sure it does not introduce its own divergences. That is, without actually

calculating the integrals

$$\iiint dx dy dz \delta(x + y + z - 1) F(x, y, z) \quad (12)$$

$$\text{and } \iiint dx dy dz \delta(x + y + z - 1) F(x, y, z) \times \log G(x, y, z; p^2/m^2) \quad (13)$$

make sure that they converge. Pay attentions to the boundaries of the parameter space and especially to the corners where $x, y \rightarrow 0$ while $z \rightarrow 1$ (or $x, z \rightarrow 0$, or $y, z \rightarrow 0$).

This calculation shows that

$$\frac{d\Sigma}{dp^2} = \frac{\text{constant}}{\epsilon} + \text{a_finite_function}(p^2) \quad (14)$$

and hence

$$\begin{aligned} \Sigma(p^2) = & (\text{a divergent constant}) + (\text{another divergent constant}) \times p^2 \\ & + \text{a_finite_function}(p^2) \end{aligned} \quad (15)$$

up to the two-loop order. In fact, this behavior persists to all loops, so all the divergences of $\Sigma(p^2)$ may be canceled with just two counterterms, δ^m and $\delta^Z \times p^2$.

6. Finally, let's use bare perturbation theory (bare λ and bare m^2 instead of the counterterms) and calculate field strength renormalization factor

$$Z = \left[1 - \frac{d\Sigma}{dp^2} \right]^{-1} \quad (16)$$

The derivative here should be evaluated at $p^2 = M_{\text{ph}}^2$ — the physical mass² of the scalar particle, but to the leading approximation we may let $M_{\text{ph}}^2 \approx m^2$ and set $p^2 = m^2$ in eq. (8). This should simplify the $G(x, y, z)$ function, but the integral is still a big mess.

Do not try to evaluate the integrals (12) and (13) by hand — it would take way too much time. Instead, use *Mathematica* or equivalent software. To help it along, replace

the (x, y, z) variables with (w, ξ) according to

$$x = \xi \times w, \quad y = (1 - \xi) \times w, \quad z = 1 - w,$$

$$\iiint_0^1 dx dy dz \delta(x + y + z - 1) = \int_0^1 dw w \int_0^1 d\xi, \quad (17)$$

then integrate over the w variable first and over the ξ second. Here is a couple of integrals I did this way you might find useful:

$$\iint \iint dx dy dz \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} = \frac{1}{2},$$

$$\iint \iint dx dy dz \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} \times \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz)^2} = -\frac{3}{4}. \quad (18)$$

Alternatively, you may evaluate the integrals like this numerically. In this case, don't bother changing variables, just use a simple 2D grid spanning a triangle defined by $x + y + z = 1$, $x, y, z \geq 0$; modern computers can sum up to 10^8 grid points in just a few seconds. But watch out for singularities at the corners of the triangle.