1. Consider the Wess–Zumino model, a QFT comprising a Majorana spinor $\Psi(x)$, a real scalar $\Phi_1(x)$, and a real pseudoscalar $\Phi_2(x)$, all massless. The Lagrangian is

$$\mathcal{L} \;=\; i\overline{\Psi}\,\partial\!\!\!/\Psi \;+\; \tfrac{1}{2}(\partial_\mu\Phi_1)^2 \;+\; \tfrac{1}{2}(\partial_\mu\Phi_2)^2 -\; g\overline{\Psi}(\Phi_1 + i\gamma^5\Phi_2)\Psi \;-\; \frac{\lambda}{8}\left(\Phi_1^2+\Phi_2^2\right)^2, \qquad (1)$$

and there is a global $U(1)$ chiral symmetry which acts as

$$\Psi \;\to\; \exp(i\theta\gamma^5)\Psi, \qquad (\Phi_1 + i\Phi_2) \;\to\; \exp(-2i\theta)\times(\Phi_1+i\Phi_2). \qquad (2)$$

Wess and Zumino found that for $\lambda = g^2$, the renormalization of this theory is particularly simple, but at first they did not know why. Salam and Strathdee realized there must be a symmetry behind this simplicity, and after working very hard to find it, they discovered the *supersymmetry*.

Thanks to the chiral symmetry, the WZ model needs only 5 independent counterterms, namely δ^g, δ^λ, δ^Z_ϕ, δ^Z_ψ, and δ^m_ϕ, but no δ^m_ψ! In general, $\delta^m_\phi = O(\Lambda^2)$ while the other counterterms are $O(\log(\Lambda/E)$.

(a) For $\lambda = g^2$, the quadratic divergence of the two-scalar 1PI amplitude vanishes. Instead, $\Sigma^{\rm loops}_\phi(p^2) = p^2\times O(\log\Lambda^2/p^2)$ and hence $\delta^m_\phi = 0$ while $\delta^Z_\phi = O(\log\Lambda/E)$. Show that this is true at the one-loop level.
Note: Feynman rules for the Majorana fermions are similar to those for the Dirac fermions (same propagators, vertices, and external leg factors), but there is an extra factor $\frac{1}{2}$ for each closed fermionic loop. (*i.e.*, $-\frac{1}{2}\,{\rm tr}(\cdots)$ instead of $-\,{\rm tr}(\cdots)$).

(b) Next, calculate the infinite parts of the other 4 counterterms at the one-loop level. Proceed similarly to homework #17, and do not hesitate to recycle similar calculations instead of redoing them from scratch. Do not assume $\lambda = g^2$ at this stage.

(c) Calculate the anomalous dimensions of the scalar and fermionic fields to order $O(g^2,\lambda)$ and show that $\gamma_\phi = \gamma_\psi$. Note: at the one-loop level this is true for any λ, but at the higher loop levels $\gamma_\phi = \gamma_\psi$ only when $\lambda = g^2$.

(d) Calculate the beta-functions $\beta_g(g,\lambda)$ and $\beta_\lambda(g,\lambda)$ to one-loop order for general λ and g. Then show that

$$\text{for } \lambda = g^2, \quad \beta_\lambda(\lambda = g^2) \; = \; 2g \times \beta_g(\lambda = g^2). \tag{3}$$

Note: because of supersymmetry, this relation holds true to all orders of the perturbation theory. But in this exercise, you should check it at the one-loop level only.

(e) Show that the relation (3) implies that **if** $\lambda(E_0) = g^2(E_0)$ for any particular energy E_0, **then** $\lambda(E) = g^2(E)$ for all energies E.

(f) Finally, consider the renormalization group flow in the (g^2, λ) plane. In the UV $\to$ IR direction, is the $\lambda = g^2$ line attractive or repulsive?

2. And now a reading assignment: *Quantum Mechanics and Path Integrals* by Feynman & Hibbs. Read all you can about care and use of Path Integrals. After the break, I will talk about "path" integrals in QFT, and it would help if you already know something about path integrals in the ordinary QM.