PHY–396 L. Problem set #20. Due April 2, 2009.

1. In the Standard Model, the electron becomes massive due to its interaction with the Higgs scalar. But the Higgs decouples below the weak scale $E \sim 100$ GeV while the electron's mass becomes important at much lower energies $E \sim m_e \sim 0.511$ MeV, and there is is a significant renormalization of the electron's mass between these two scales, mostly due to EM interactions. So in this exercise, we consider the renormalization of the electron mass in pure QED.

   (a) Show that for $E \gg m_e$ the running electron mass satisfies

   $$\frac{dm_e(E)}{d\log E} = m_e(E) \times \gamma_m(e^2) \tag{1}$$

   and write a formula for the anomalous dimension γ_m of the mass in terms of the counterterms $\delta_2(E)$ and $\delta_m(E)$.

   (b) Calculate the infinite parts of the counterterms $\delta_2(E)$ and $\delta_m(E)$ to one-loop order. Instead of the Feynman gauge used in earlier homeworks, please use the generic Lorentz-invariant gauge

   

   $$= \frac{-i}{k^2 + i0} \times \left(g^{\mu\nu} + (\xi - 1)\frac{k^\mu k^\nu}{k^2} \right). \tag{2}$$

   Note that both counterterms in question are gauge-dependent.

   (c) The γ_m should be gauge invariant. Calculate it to one-loop order and show that the gauge-dependence (or at least ξ-dependence) does cancel out.

   (d) Given $m_e(E) = m_e^{\text{phys}} = 511$ keV for $E = m_e^{\text{phys}}$, calculate the running electron mass $m_e(E)$ at the weak scale.

2. Consider a particle on a 1D circle of radius R, or equivalently a 1D particle in a box of length $L = 2\pi R$ with periodic boundary conditions where moving past the $X = L$ points brings you back to $x = 0$. In other words, the particle's position $x(t)$ is defined modulo L.

(a) Consider all possible particle's paths from a fixed point x_0 (modulo L) at time $t = 0$ to a fixed point x' (modulo L) at time $t = T$. Show that the space of such paths is isomorphic to the space of free particle's paths from a fixed x_0 at $t = 0$ to any of the points $x' + nL$ at $t = T$, for all integer $n = 0, \pm 1, \pm 2, \ldots$. Then use path integral formalism to show that

$$U_{\rm box}(x'; x_0) \;=\; \sum_{n=-\infty}^{+\infty} U_{\rm free}(x' + nL; x_0) \tag{3}$$

where $U_{\rm box}$ and $U_{\rm free}$ are the evolution kernels (between times $t = 0$ and $t = T$) for the particle in a box and for the free particle.

According to Poisson re-summation formula, if a function $F(n)$ of integer n can be continued to a function $F(\nu)$ of arbitrary real ν, then

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} F(n) \;&=\; \int\! d\nu\, F(\nu) \times \sum_{n=-\infty}^{+\infty} \delta(\nu - n) \\ &=\; \sum_{\ell=-\infty}^{+\infty} \int\! d\nu\, F(\nu) \times e^{2\pi i \ell \nu}. \end{aligned} \tag{4}$$

(b) The free particle (living on an infinite 1D line) has evolution kernel

$$U_{\rm free}(x'; x_0) \;=\; \sqrt{\frac{M}{2\pi i \hbar T}} \times \exp\left(+\frac{iM(x' - x_0)^2}{2\hbar T}\right). \tag{5}$$

Plug this free kernel into eq. (3) and use Poisson formula to sum over n.

(c) Verify that the resulting evolution kernel for the particle in a box agrees with the usual QM formula

$$U_{\rm box}(x'; x_0) \;=\; \sum_p L^{-1/2} e^{ipx'/\hbar} \times e^{-iT(p^2/2M)/\hbar} \times L^{-1/2} e^{-ipx_0/\hbar} \tag{6}$$

where p takes box-quantized values

$$p \;=\; \frac{2\pi\hbar}{L} \times \text{integer}. \tag{7}$$

3. Finally, a reading assignment: Please read my notes about path integral of the harmonic oscillator. Pay particular attention to the first coupe of pages about the Euclidean path integral, but read the rest of the notes too.