1. First, a bit of group theory. Consider a generic simple non-abelian compact Lie group G and its generators T^a. For a suitable normalization of the generators,

$$\mathrm{tr}_{(r)}(T^aT^b) \ \equiv\ \mathrm{tr}\left(T^a_{(r)}T^b_{(r)}\right) \ =\ R(r)\delta^{ab} \tag{1}$$

where the trace is taken over any complete multiplet (r) — irreducible or reducible, it does not matter — and $T^a_{(r)}$ is the matrix representing the generator T^a in that multiplet. The coefficient $R(r)$ in eq. (1) depends on the multiplet (r) but it's the same for all generators T^a and T^b.

The (quadratic) Casimir operator $C_2 = \sum_a T^aT^a$ commutes with all the generators, $\forall b$, $[C_2, T^b] = 0$. Consequently, when we restrict this operator to any *irreducible* multiplet (r) of the group G it becomes a unit matrix times some number $C(r)$. In other words,

$$\text{for an irreducible } (r),\quad \sum_a T^a_{(r)}T^a_{(r)} \ =\ C(r)\times \mathbf{1}_{(r)}\,. \tag{2}$$

For example, for the isospin group $SU(2)$, the Casimir operator is $C_2 = \vec{I}^2$, the irreducible multiplets have definite isospin $I = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and $C(I) = I(I+1)$.

(a) Show that for any irreducible multiplet (r),

$$\frac{R(r)}{C(r)} \ =\ \frac{\dim(r)}{\dim(G)}\,. \tag{3}$$

In particular, for the $SU(2)$ group, this formula gives $R(I) = \frac{1}{3}I(I+1)(2I+1)$.

(b) Suppose the first three generators of G generate an $SU(2)$ subgroup. Show that if a multiplet (r) of G decomposes into several $SU(2)$ multiplets of isospins $I_1, I_2, \ldots, I_n$, then

$$R(r) \ = \sum_{i=1}^{n} \tfrac{1}{3}I_i(I_i+1)(2I_i+1). \tag{4}$$

(c) Now consider the $SU(N)$ group with an obvious $SU(2)$ subgroup of matrices acting only on the first two components of a complex N-vector. This complex N-vector is called the fundamental multiplet (of the $SU(N)$) and denoted (N) or $\mathbf{N}$. As far as the $SU(2)$ subgroup is concerned, (N) comprises one doubles and $N-2$ singlets, hence hence

$$R(N) \;=\; \tfrac{1}{2} \qquad \text{and} \quad C(N) \;=\; \frac{N^2-1}{2N}\,. \tag{5}$$

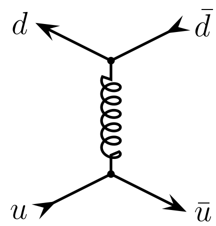
Show that the adjoint multiplet of the $SU(N)$ decomposes into one $SU(2)$ triplet, $2(N-2)$ doublets, and $(N-2)^2$ singlets, therefore

$$R(\text{adj}) \;=\; C(\text{adj}) \;\equiv\; C(G) \;=\; N. \tag{6}$$

Hint: $(N)\times(\overline{N}) = (\text{adj}) + (1)$.

(d) The symmetric and the anti-symmetric 2–index tensors form irreducible multiplets of the $SU(N)$ group. Find out the decomposition of these multiplets under the $SU(2)\subset SU(N)$ and calculate their respective R factors and Casimirs C.

2. Now let's apply this group theory to physics. Consider quark-antiquark pair production in QCD, specifically $u\bar{u}\to d\bar{d}$. There is only one tree diagram contributing to this process,



(7)

Evaluate this diagram, then sum/average the $|\mathcal{M}|^2$ over both spins and *colors* of the final/initial particles to calculate the total cross section. For simplicity, you may neglect the quark masses.

Note that the diagram (7) looks exactly like the QED pair production process $e^-e^+\to$ virtual $\gamma\to\mu^-\mu^+$, so you can re-use the QED formula for summing/averaging over the spins. But in QCD, you should also sum/average over colors of all the quarks, and that's the whole point of this exercise.

3. Next, consider a scalar analogue of QCD, or more generally a theory of Yang–Mills fields A^a_μ and complex scalars Φ_i in some multiplet (r) of the gauge group G.

   (a) Write down the Lagrangian and the Feynman rules of this theory.

   Next, consider the annihilation process $\Phi + \Phi^* \to 2$ gauge bosons. At the tree level, there are four Feynman diagrams contributing to this process.

   (b) Draw the diagrams and write down the tree-level annihilation amplitude.

   As discussed in class, amplitudes involving the non-abelian gauge fields satisfy a weak form of the Ward identity: *On-shell Amplitudes involving* **a** *longitudinally polarized gauge boson vanish, provided all the other gauge bosons are transversely polarized.* In other words,

$$\mathcal{M} \;\equiv\; e_1^{\mu_1} e_2^{\mu_2} \cdots e_n^{\mu_n}\, \mathcal{M}_{\mu_1\mu_2\cdots\mu_n}(\text{momenta}) \;=\; 0$$
$$\text{when}\quad e_1^\mu \propto k_1^\mu \quad \text{but} \quad e_2^\nu k_{2\nu} \;=\; \cdots \;=\; e_n^\nu k_{n\nu} \;=\; 0.$$

   (c) Verify this identity for the scalar annihilation amplitude.

4. To convert the annihilation amplitude into a cross-section we need to sum/average over the colors of all the particles. As a first step in this direction, it's convenient to write the amplitude as

$$\mathcal{M}(j + i \to a + b) \;=\; F \times \{T^a, T^b\}^i{}_j \;+\; iG \times [T^a, T^b]^i{}_j \tag{8}$$

   where F and G are some functions of momenta and polarizations of the vector particles while a, b, i, and j are the color indices of the four particles. Specifically, the a and b colors of the gauge bosons run over the adjoint multiplet of G, the j index of the scalar 'quark' runs over the multiplet (r), and the i index of the scalar 'antiquark' runs over the conjugate multiplet $(\bar{r})$.

   (a) Show that the annihilation amplitude indeed has form (8) and write down the coefficients F and G as explicit functions of the particles momenta and polarizations.

(b) Next, let us sum the $|\mathcal{M}|^2$ over the gauge boson's colors and average over the scalars' colors. Show that

$$\frac{1}{\dim^2(r)} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 \;=\; \frac{C(r)}{\dim(r)} \times \left(4C(r) \times |F|^2 \,+\, C(\mathrm{adj}) \times (|G|^2 - |F|^2)\right). \quad (9)$$

In particular, for scalars in the fundamental representation of the $SU(N)$ gauge group,

$$\frac{1}{N^2} \sum_{ij} \sum_{ab} |\mathcal{M}|^2 \;=\; \frac{N^2-1}{2N^2} \left(\frac{N^2-2}{N}\, |F|^2 \,+\, N|G|^2\right). \quad (10)$$

(c) Evaluate F and G in the center of mass frame. In this frame, the vector particles' polarizations $e^{\mu}_{1,2} = (0, \mathbf{e}_{1,2})$ are purely spatial and transverse to the vectors momenta $\pm\mathbf{k}$. For simplicity, use planar rather than circular polarizations.

(d) Finally, calculate the (polarized, partial) cross-section for the annihilation process.