1. Let's start with the pion decay. In class I have explained how the QED anomaly of the axial quark current makes the neutral pion decay into two photons, $\pi^0 \to \gamma\gamma$.

   (a) Finish the calculation of the neutral pion's lifetime. For your information, pion decay constant is $f_\pi \approx 93$ MeV and the mass of π^0 is about 135 MeV.

   The f_π is called the *pion decay constant* because it controls the decay of the charged pions into muons and neutrinos, $\pi^+ \to \mu^+\nu_\mu$ and $\pi^- \to \mu^-\bar{\nu}_\mu$. This decay is is due to weak interaction, and since $M_\pi \ll M_W$ we may use Fermi's current-current interactions

$$\mathcal{L} = -2\sqrt{2}G_F J_L^{+\alpha} J_{L\alpha}^{-} \quad \text{where } J_L^{\pm\alpha} = \tfrac{1}{2}\bigl(J_V^{\pm\alpha} - J_A^{\pm\alpha}\bigr). \tag{1}$$

   For the pion decay process, one of the currents annihilates the charged pion while the other creates the lepton pair.

   (b) Show that the decay amplitude is

$$\mathcal{M}(\pi^+ \to \mu^+\nu_\mu) = G_f f_\pi p^\alpha(\pi) \times \bar{v}(\mu)(1-\gamma^5)\gamma_\alpha u(\nu). \tag{2}$$

   (c) Sum over the fermion spins and calculate the decay rate $\Gamma(\pi^+ \to \mu^+\nu_\mu)$. Show that this decay rate would vanish if both leptons are massless, and that's why the pion decays mostly into $\mu^+\nu_\mu$ rather then $e^+\nu_e$. Specifically,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{M_e^2}{M_\mu^2}\,\frac{(1-(M_e/M_\pi)^2)^2}{(1-(M_\mu/M_\pi)^2)^2} \approx 1.2\cdot 10^{-4}. \tag{3}$$

   (d) Explain this preference for the heavier lepton in terms of mis-match between chirality and helicity.

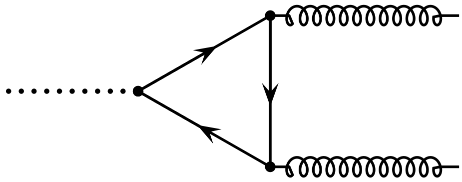
2. Now consider the axial anomaly in a non-abelian gauge theory, for example QCD with one massless quark flavor. In this case,

$$\partial_\mu\Big(J^{5\mu}\ =\ \overline{\Psi}_i\gamma^5\gamma^\mu\Psi^i\Big)\ =\ \frac{g^2}{16\pi^2}\,\epsilon^{\alpha\beta\gamma\delta}\,\mathrm{tr}\Big(F_{\alpha\beta}F_{\gamma\delta}\Big) \tag{4}$$

where $F_{\mu\nu}$ is the non-abelian gauge field strength.

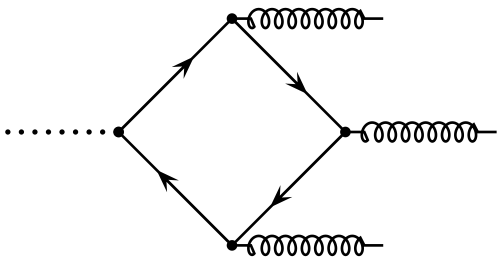
(a) Expand the right hand side of eq. (4) into 2–gluon, 3–gluon, and 4–gluon terms and show that the 4–gluon term vanishes identically.
Hint: Use the cyclic symmetry of the trace.

The two-gluon anomaly term obtains from the triangle diagrams



+ gluon permutation. (5)

This works exactly as discussed in class for the QED, except in QCD we should trace $F_{\alpha\beta}F_{\gamma\delta}$ over the quark colors. But in QCD there is also the three-gluon anomaly (*cf.* part (a)) which obtains from the quadrangle diagrams



+ gluon permutations. (6)

(b) Evaluate the quadrangle diagrams using the Pauli–Villars regularization and derive the three-gluon anomaly in QCD.

3. In any *even* spacetime dimension $d = 2n$, a massless Dirac fermion has an axial symmetry $\Psi(x) \to \exp(i\theta\Gamma)\Psi(x)$ where Γ generalizes the γ^5. Classically, the axial current $J^\mu_A = \overline{\Psi}\Gamma\gamma^\mu\Psi$ is conserved, but when the fermion is coupled to a gauge field — abelian or non-abelian — the axial symmetry is broken by the anomaly and

$$\partial_\mu J^\mu_A \;=\; -\frac{2}{n!}\left(\frac{g}{4\pi}\right)^n \epsilon^{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}\,\mathrm{tr}\Big(F_{\alpha_1\beta_1}F_{\alpha_2\beta_2}\cdots F_{\alpha_n\beta_n}\Big). \tag{7}$$

In class, we have derived this formula for $d = 4$ by formally calculating the Jacobian $\mathrm{Det}(2i\theta\gamma^5)$ of the fermionic path integral for the axial symmetry. In this exercise, you should similarly derive the anomaly equation (7) for any even spacetime dimension $d = 2n$.

For your information, in $2n$ Euclidean dimensions $\{\gamma^\mu,\gamma^\nu\} = +2\delta^{\mu\nu}$, $\Gamma = i^{n-2}\gamma^1\gamma^2\cdots\gamma^{2n}$, $\{\Gamma,\gamma^\mu\} = 0$, $\Gamma^2 = +1$, and $\mathrm{tr}(\Gamma\gamma^\alpha\gamma^\beta\cdots\gamma^\omega) = 2^n i^{2-n}\epsilon^{\alpha\beta\cdots\omega}$ (for $2n = d$ matrices $\gamma^\alpha\cdots\gamma^\omega$).