

Magnetic Monopoles.

The easiest way to visualize a magnetic monopole is by considering a pole of a long, thin magnet or an end point of a long, thin solenoid. Let us choose our coordinates such that the pole is at the origin and the magnet goes along the negative z semiaxis. Then, in polar coordinates,

$$\mathbf{B}(r, \theta, \phi) = \frac{m}{r^2} \mathbf{e}_r \quad (1)$$

(\mathbf{e}_r us the unit vector in the radial direction) everywhere outside the magnet, while inside the magnet there is magnetic flux $4\pi m$ towards the pole.

Suppose the magnet is infinitely thin, infinitely long and does not interact with the rest of the universe except through the magnetic field it carries. Classically, all one can observe under such circumstances is the magnetic field (1), so for all intents and purposes we have a magnetic monopole of magnetic charge m . In quantum mechanics however, one can also detect the Aharonov-Bohm effect due to the magnetic flux $4\pi m$ inside the magnet. This makes the magnet itself detectable along its entire length — unless $4\pi m \cdot q/2\pi\hbar c$ is an integer and the Aharonov-Bohm effect disappears. Hence, the Dirac's quantization condition: *For all magnetic monopoles in the universe and for all electrically-charged particles in the universe,*

$$mq = \left(\frac{1}{2}\hbar c\right) \text{ times an integer.} \quad (2)$$

Consequently, *if there is a magnetic monopole anywhere in the universe, all electrical charges must be quantized.*

A more rigorous argument (made by P. A. M. Dirac himself years before the discovery of the Aharonov-Bohm effect) involves two singular vector potentials

$$\mathbf{A}_{\pm}(r, \theta, \phi) = m \frac{\pm 1 - \cos \theta}{r \sin \theta} \mathbf{e}_{\phi}, \quad (3)$$

which are gauge-equivalent to each other:

$$\mathbf{A}_+ - \mathbf{A}_- = \frac{2m \mathbf{e}_\phi}{r \sin \theta} = 2m \nabla \phi \quad (4)$$

and thus lead to the same magnetic field \mathbf{B} , namely (1):

$$\begin{aligned} \nabla \times \mathbf{A}_\pm &= \nabla \times (m(\pm 1 - \cos \theta) \nabla \phi) = m (\nabla(\pm 1 - \cos \theta)) \times \nabla \phi \\ &= m \frac{\sin \theta \mathbf{e}_\theta}{r} \times \frac{\mathbf{e}_\phi}{r \sin \theta} = \frac{m \mathbf{e}_r}{r^2}. \end{aligned} \quad (5)$$

The vector potential \mathbf{A}_+ has a so-called ‘‘Dirac string’’ of singularities along the negative z semiaxis ($\theta = \pi$); similarly, the \mathbf{A}_- has a Dirac string along the positive z semiaxis ($\theta = 0$). Although both the \mathbf{A}_+ and the \mathbf{A}_- are singular, together they provide for a non-singular picture of the monopole’s field: One simply uses the \mathbf{A}_+ in the region $0 \leq \theta < \pi - \epsilon$ (where it’s non-singular) while the \mathbf{A}_- should be used for $\epsilon < \theta \leq \pi$; together, the two regions cover the whole space (except the site of the monopole itself). In the overlap region $\epsilon < \theta < \pi - \epsilon$ one may use either \mathbf{A}_+ or \mathbf{A}_- .

In quantum mechanics, a gauge transformation of the vector potential \mathbf{A} should be accompanied by a phase transformation of the wave functions of charged particles. Hence, using two gauge-equivalent vector potentials to describe the monopole’s field requires us to use two wave functions for any charged particle, namely $\Psi_+(r, \theta, \phi)$ for $0 \leq \theta < \pi - \epsilon$ and $\Psi_-(r, \theta, \phi)$ for $\epsilon < \theta \leq \pi$. In the overlap region $\epsilon < \theta < \pi - \epsilon$, the two wave functions must be related to each other by a phase transformation that corresponds to the gauge transformation (4):

$$\Psi_+(r, \theta, \phi) = \Psi_-(r, \theta, \phi) \cdot \exp\left(\frac{2iqm\phi}{\hbar c}\right). \quad (6)$$

In order to make physical sense, a wave-function of a particle must be continuous and single-valued. In the presence of a monopole, *both* Ψ_+ and Ψ_- must be single-valued. Hence, the relation (6) makes physical sense if and only if $\exp(4\pi i q m / \hbar c) = 1$, *i.e.* the product of the monopole’s magnetic charge m and the particle’s electric charge q satisfies the Dirac’s quantization condition (2).

In gaussian units of measurement, magnetic and electric charges have the same dimensionality. However, the quanta of the two charges are quite different: The electric charges of all free particles are quantized in units of e ; hence, according to eq. (2), all the magnetic charges should be quantized in units of

$$\frac{\hbar c}{2e} \approx \frac{137}{2}e. \quad (7)$$

Of course, as far as the Quantum electrodynamics is concerned, the monopoles do not have to exist at all, but if they do exist, their charges must be quantized in units of (7). Furthermore, if as much as one magnetic monopole exist anywhere in the universe then the electric charges of all free particles must be exactly quantized. Historically, Dirac discovered the magnetic monopole while trying to explain *the value* of the electric charge quantum e ; instead, he found a reason for the charge quantization, but no explanation for $e^2 \approx \hbar c/137$, and he was quite disappointed.

Today, we have other explanations of the electric charge quantization; in particular the Grand Unification of strong, weak and electromagnetic interactions at extremely high energies produces quantized electrical charges. Curiously, the same Grand Unified Theories also predict that there **are** magnetic monopoles with charges (7). More recently, several attempts to unify all the fundamental interactions within the context of the String Theory also gave rise to magnetic monopoles, with charges quantized in units of $N\hbar c/2e$, where N is an integer such as 3 or 5. It was later found that in the same theories, there were superheavy particles with fractional electric charges e/N , so the monopoles in fact had the smallest non-zero charges allowed by the Dirac condition (2)! Presently, some theoretical physicists believe that any fundamental theory that provides for exact quantization of the electric charge should also provide for the existence of magnetic monopoles, but this conjecture has not been proven (yet).

Suggested Reading: J. J. Sakurai, *Modern Quantum Mechanics*, §2.6.