Introduction to Path Integrals

Consider ordinary quantum mechanics of a single particle in one space dimension. Let's work in the coordinate space and study the evolution kernel

$$U(t_B,x_B;T_A,x_A) \;=\; \langle x_B|\, e^{-i(t_B-t_A)\hat{H}/\hbar}\, |x_A\rangle \tag{1}$$

— the amplitude for moving from point x_A at time t_A to point x_B at time t_B. In the semi-classical regime, this kernel is given by the WKB approximation

$$U(B;A) \;\approx\; \text{prefactor} \times \exp(iS[x_{\rm cl}(t)]/\hbar) \tag{2}$$

where

$$S[x(t)] \;=\; \int\limits_{t_A}^{t_B} L(x(t),\dot{x}(t))\,dt \tag{3}$$

is the action integral of the *classical mechanics* and $x_{\rm cl}(t)$ is the classical path from A to B — *i.e.*, $x_{\rm cl}(t)$ is the minimum of the functional $S[x(t)]$ under conditions $x(t_A)=x_A$ and $x(t_B)=x_B$. If there are several classical paths from A to B because $S[x]$ has several local minima, then they all contribute to the kernel with appropriate phases and we get interference,

$$U(B;A) \;\approx\; \sum_{\substack{\text{classical}\\ \text{paths } i}} \text{prefactor}_i \times \exp(iS[x_i(t)]/\hbar). \tag{4}$$

In the exact quantum mechanics, a sum (4) over classical path becomes an integral over all possible path from A to B,

$$U(B;A) \;=\; \iiint\limits_{x(t_A)=x_A}^{x(t_B)=x_B} \mathcal{D}[x(t)]\, \exp\bigl(iS[x(t)]/\hbar\bigr). \tag{5}$$

Note that we integrate over all differentiable functions $x(t)$ that satisfy the boundary condition at t_A and t_B, and they do not obey any equations of motion except by accident. However, in the semiclassical $\hbar\to 0$ limit, contribution of most paths is washed out by interference with similar paths whose action differs by only $O(\hbar)$. The only survivors of this wash-out stationary "points" of the functional $S[x(t)]$, which are precisely the classical paths from A to B. This is how the WKB approximation (4) — and eventually the classical mechanics — emerge in the $\hbar \to 0$ limit.

The problem with the *path integral* (5) is how to define the differential $\mathcal{D}[x(t)]$ of a path. Obviously, we should discretize the time: Slice a continuous time interval $t_A \leq t \leq t_B$ into a large but finite set of discrete times

$$(t_0, t_1, t_2, \ldots, t_{N-1}, t_N), \quad t_n = t_A + n\Delta t, \quad \Delta t = \frac{t_B - t_A}{N}, \quad t_0 = t_A, \quad t_N = t_B, \quad (6)$$

but eventually take the $N \to \infty$ limit. This gives us

$$\mathcal{D}[x(t)] \stackrel{\text{def}}{=} \lim_{N\to\infty} dx_1\, dx_2 \cdots dx_{N-1} \times \text{normalization\_factor} \qquad \text{where} \quad x_n \equiv x(t_n). \quad (7)$$

Note that we do not integrate over the $x_0 \equiv x(t_A)$ and $x_N \equiv x(t_B)$ because they are fixed by the boundary conditions in eq. (5).

The non-obvious part of eq. (7) is the normalization_factor. We shall see later in these notes that this factor depends on N, on the net time $T = t_B - t_A$, and even on the particle's mass, and the exact formula for this factor is not easy to guess. Fortunately, there is a different version of path integration that does not suffer from such normalization factors.

Let's consider paths in the phase space (x, p) rather than just the x-space. In other words, let's treat $x(t)$ and $p(t)$ as independent variables and write the action integral (3) in the Hamiltonian language

$$S[x(t), p(t)] = \int_A^B \big[p\, dx - H(x(t), p(t))\, dt\big] \quad (8)$$

as a functional of both $x(t)$ and $p(t)$. A classical path is a minimax of this functional — a (local) minimum with respect to variations of $x(t)$ but a (local) maximum with respect to variations of $p(x)$. Also, the coordinate $x(t)$ is subject to boundary conditions at the start A

and finish B, but there are no boundary conditions for the momentum $p(t)$. In the quantum mechanics,

$$U(B;A) \;=\; \iiint\limits_{x(t_A)=x_A}^{x(t_B)=x_B} \mathcal{D}'[x(t)] \iiint \mathcal{D}[p(t)] \, \exp\bigl(iS[x(t),p(t)]/\hbar\bigr) \tag{9}$$

where

$$\mathcal{D}'[x(t)] \times \mathcal{D}[p(t)] \;=\; \lim_{N\to\infty} \prod_{n=1}^{N-1} dx_n \times \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar}\,. \tag{10}$$

This time, there are no funny normalization factors: all we have is $1/2\pi\hbar$ for every dp_n, and that's standard procedure in quantum mechanics. Note that for a given N, we integrate over N momenta but only $N-1$ coordinates because of the boundary conditions on both ends; to make this difference explicit, I have marked the $\mathcal{D}'[x(t)]$ with a prime.

Deriving the Phase–Space Path Integral from the Hamiltonian QM

Let's start with a mathematical **lemma:**

$$\lim_{N\to\infty} \left(e^{\hat{a}/N} \times e^{\hat{b}/N}\right)^N \;=\; e^{\hat{a}+\hat{b}} \tag{11}$$

even if the operators $\hat{a}$ and $\hat{b}$ do not commute with each other. **Proof:**

$$e^{\hat{a}/N} \times e^{\hat{b}/N} \;=\; 1 \;+\; \frac{\hat{a}+\hat{b}}{N} \;+\; O(1/N^2), \tag{12}$$

and

$$\lim_{N\to\infty} \left(1 \;+\; \frac{\hat{a}+\hat{b}}{N} \;+\; O(1/N^2)\right)^N \;=\; e^{\hat{a}+\hat{b}} \tag{13}$$

regardless of the details of the $O(1/N^2)$ terms.

Now consider a quantum particle living in one space dimension with a Hamiltonian operator of the form

$$\hat{H} \;=\; K(\hat{p}) \;+\; V(\hat{x}) \tag{14}$$

where the kinetic energy $\hat{K} \equiv K(\hat{p})$ does not depend on the position $\hat{x}$ and the potential energy $\hat{V} = V(\hat{x})$ does not depend on the momentum $\hat{p}$. Using the lemma (11), we may write the evolution operator for the particle as

$$\hat{U}(t_B - t_A) \;\equiv\; e^{-i\hat{H}(t_B-t_A)/\hbar} \;=\; \lim_{N\to\infty}\left(e^{-i\hat{V}\Delta t/\hbar}\times e^{-i\hat{K}\Delta t/\hbar}\right)^N \tag{15}$$

where $\Delta t = (t_B - t_A)/N$ as in eq. (6). Consequently, in the coordinate basis

$$\langle x_B|\,\hat{U}(t_B - t_A)\,|x_A\rangle \;=\; \lim_{N\to\infty}\int\!dx_1\cdots\int\!dx_{N-1}\prod_{n=1}^{N}\langle x_n|\,e^{-i\hat{V}\Delta t/\hbar}\times e^{-i\hat{K}\Delta t/\hbar}\,|x_{n-1}\rangle \tag{16}$$

where we have identified $x_0 \equiv x_A$ and $x_N \equiv x_B$. Each Dirac bracket in the above product evaluates to

$$\begin{aligned}
&\langle x_n|\,e^{-i\hat{V}\Delta t/\hbar}\times e^{-i\hat{K}\Delta t/\hbar}\,|x_{n-1}\rangle \;= \\
&\quad=\; e^{-iV(x_n)\Delta t/\hbar}\times\langle x_n|\,e^{-i\hat{K}\Delta t/\hbar}\,|x_{n-1}\rangle \\
&\quad=\; e^{-iV(x_n)\Delta t/\hbar}\times\int\!\frac{dp_n}{2\pi\hbar}\,\langle x_n|p_n\rangle\,e^{-iK(p_n)\Delta t/\hbar}\,\langle p_n|x_{n-1}\rangle \\
&\quad=\int\!\frac{dp_n}{2\pi\hbar}\,e^{-iV(x_n)\Delta t/\hbar}\times e^{ix_np_n/\hbar}\times e^{-iK(p_n)\Delta t/\hbar}\times e^{-ix_{n-1}p_n/\hbar} \\
&\quad=\int\!\frac{dp_n}{2\pi\hbar}\,\exp\left[\frac{i}{\hbar}\Big(p_n(x_n - x_{n-1})\;-\;V(x_n)\Delta t\;-\;K(p_n)\Delta t\Big)\right].
\end{aligned} \tag{17}$$

Plugging this formula back into eq. (16) and combining all the exponentials, we arrive at

$$U(B;A) \;=\; \lim_{N\to\infty}\int\!dx_1\cdots\int\!dx_{N-1}\int\!\frac{dp_1}{2\pi\hbar}\cdots\int\!\frac{dp_N}{2\pi\hbar}\,e^{iS/\hbar}, \tag{18}$$

where

$$S \;=\; \sum_{n=1}^{N}p_n(x_n - x_{n-1}) \;-\; \Delta t\times\sum_{n=1}^{N}\Big(V(x_n)\;+\;K(p_n)\Big) \tag{19}$$

is the discretized action for a discretized path. Indeed, in the large N limit

$$\sum_{n=1}^{N}\Big[p_n\times(x_n - x_{n-1}) + (V(x_n) + K(p_n))\times\Delta t\Big] \;\xrightarrow[N\to\infty]{}\; \int_A^B\Big[p\,dx\;-\;H\,dt\Big] \;\equiv\; S[x(t),p(t)]. \tag{20}$$

Consequently, we should interpret the product of coordinate and momentum integrals in eq. (18) as the discretized integral over the paths in the momentum space,

$$\int dx_1 \cdots \int dx_{N-1} \int \frac{dp_1}{2\pi\hbar} \cdots \int \frac{dp_N}{2\pi\hbar} \xrightarrow[N\to\infty]{} \iiint \mathcal{D}'[x(t)] \iiint \mathcal{D}[p(t)] \tag{21}$$

in perfect agreement with eq. (10). And eq. (18) itself is the proof of the path-integral formula

$$U(B;A) = \iiint_{x(t_A)=x_A}^{x(t_B)=x_B} \mathcal{D}'[x(t)] \iiint \mathcal{D}[p(t)] \exp\bigl(iS[x(t),p(t)]/\hbar\bigr). \tag{9}$$

A note on discretization. Interpreting the sum $\sum_n p_n(x_n - x_{n-1})$ as the discretized integral $\int p dx$ calls for assigning the momenta p_n to mid-point discrete times with respect to the coordinates x_n:

$$x_n \equiv x(t = t_A + n\Delta t) \quad \textbf{but} \quad p_n \equiv p(t = t_A + (n - \tfrac{1}{2})\Delta t). \tag{22}$$

As long as the Hamiltonian can be split into separate kinetic and potential energies according to eq. (14), such different discrete times for the x_n and p_n are OK because

$$\int H(x,p)\,dt = \int V(x)\,dt + \int K(p)\,dt \to \Delta t \sum_{n=1}^{N} V(x_n) + \Delta t \sum_{n=1}^{N} K(p_n) \tag{23}$$

and the details of the discretization do not matter in the large N limit. However, *when the classical Hamiltonian is more complicated than a sum of kinetic and potential energies, the path integral formalism suffers from the discretization ambiguity.* For example, for

$$H(x,p) = \frac{p^2}{2M(x)} \tag{24}$$

we could discretize the action as

$$\begin{aligned} S \;\to&\; \sum_n p_n(x_n - x_{n-1}) - \Delta t \sum_n \frac{p_n^2}{2M(x_n)}, \\ \text{or} \to&\; \sum_n p_n(x_n - x_{n-1}) - \Delta t \sum_n \frac{p_n^2}{2M(x_{n-1})}, \\ \text{or} \to&\; \sum_n p_n(x_n - x_{n-1}) - \Delta t \sum_n \frac{p_n^2}{M(x_n) + M(x_{n-1})}, \\ \text{or} \to&\; \text{something else,} \end{aligned} \tag{25}$$

all these options lead to different evolution kernels, and there are no general rules how to resolve such ambiguities. In fact, *the discretization ambiguities of the path-integral formalism correspond to the operator-ordering ambiguities of the Hilbert-space formalism of quantum mechanics.* For example, given the classical Hamiltonian of the form (24), we can take the quantum Hamiltonian operators to be

$$\begin{aligned} \hat{H} &= \frac{1}{2M(\hat{x})}\,\hat{p}^2, \quad \text{or } \hat{H} = \hat{p}^2\,\frac{1}{2M(\hat{x})}\,, \quad \text{or } \hat{H} = \hat{p}\,\frac{1}{2M(\hat{x})}\hat{p}\,, \quad \text{or} \\ \hat{H} &= \frac{1}{2M(\hat{x})}\,\hat{p}M(\hat{x})\hat{p}\,\frac{1}{M(\hat{x})}\,, \quad \text{or something else?} \end{aligned} \tag{26}$$

Lagrangian Path Integral

In this section, I shall reduce the Hamiltonian path integrals over both $x(t)$ and $p(t)$ to the Lagrangian path integrals over the $x(t)$ alone by integrating over the paths in momentum space. *This works only when the kinetic energy is quadratic in the momentum,*

$$H(p,x) \;=\; \frac{p^2}{2M} \;+\; V(x) \quad\Longrightarrow\quad \hat{H} \;=\; \frac{\hat{p}^2}{2M} \;+\; V(\hat{x}). \tag{27}$$

For such Hamiltonians,

$$p\dot{x} - H(p,x) \;=\; p\dot{x} - \frac{p^2}{2M} - V(x) \;=\; -\frac{(p-M\dot{x})^2}{2M} + \frac{M\dot{x}^2}{2} - V(x) \;=\; L(\dot{x},x) - \frac{(p-M\dot{x})^2}{2M} \tag{28}$$

and consequently

$$S^{\rm Ham}[x(t),p(t)] \;=\; S^{\rm Lagr}[x(t)] \;-\; \frac{1}{2M}\int\! dt\,(p-M\dot{x})^2\,. \tag{29}$$

Therefore, in the path integral formalism,

$$\begin{aligned} U(B;A) &= \iiint_A^B \mathcal{D}'[x(t)] \iiint \mathcal{D}[p(t)]\, \exp\left(\frac{i}{\hbar}\,S^{\rm Ham}[x(t),p(t)]\right) \\ &= \iiint_A^B \mathcal{D}'[x(t)]\, \exp\left(\frac{i}{\hbar}\,S^{\rm Lagr}[x(t)]\right) \times \iiint \mathcal{D}[p(t)]\, \exp\left(\frac{-i}{2M\hbar}\int\! dt\,(p-M\dot{x})^2\right). \end{aligned} \tag{30}$$

On the second line here, we integrate over the coordinate-space paths $x(t)$ after integration over the momentum-space paths $p(t)$, so as far as $\iiint\mathcal{D}[p(t)]$ is concerned, we can treat the coordinate-space path $x(t)$ as a constant. Also, the path-integral measure is linear so we may shift the integration variable by a constant, thus

$$\begin{aligned}
\iiint\mathcal{D}[p(t)]\,\exp\left(\frac{-i}{2M\hbar}\int\!dt\,(p-M\dot{x})^2\right) &= \iiint\mathcal{D}[p(t)-M\dot{x}(t)]\,\exp\left(\frac{-i}{2M\hbar}\int\!dt\,(p-M\dot{x})^2\right)\\
&= \iiint\mathcal{D}[p'(t)]\,\exp\left(\frac{-i}{2M\hbar}\int\!dt\,p'^2(t)\right)\\
&= \text{const.}
\end{aligned}\tag{31}$$

Plugging this formula back into eq. (30) gives us a Lagrangian path integral

$$U(B;A)\;=\;\text{const}\times\iiint_{x(t_A)=x_A}^{x(t_B)=x_B}\mathcal{D}'[x(t)]\,\exp\left(\frac{i}{\hbar}\,S^{\text{Lagr}}[x(t)]\right).\tag{32}$$

In this formalism there is no independent momentum-space path $p(t)$, we integrate only over the coordinate-space path $x(t)$, and the action is given by the Lagrangian formula (3). However, the price of this simplification is the un-known overall constant multiplying the path integral (32).

To calculate this constant we should first discretize time and only then integrate out the discrete momenta p_n. For finite N, the discretized Hamiltonian-formalism action (19) can be written as

$$\begin{aligned}
S^{\text{Ham}}_{\text{discr}}(x_0,\ldots,x_N;p_1,\ldots,p_N) &= \sum_n p_n(x_n-x_{n-1})\;-\;\frac{\Delta t}{2M}\sum_n p_n^2\;-\;\Delta t\sum_n V(x_n)\\
&= -\frac{\Delta t}{2M}\sum_n\left(p_n\;-\;M\frac{x_n-x_{n-1}}{\Delta t}\right)^2\\
&\quad+\;\frac{M}{2\Delta t}\sum_n(x_n-x_{n-1})^2\;-\;\Delta t\sum_n V(x_n)\\
&= -\frac{\Delta t}{2M}\sum_n\left(p_n\;-\;M\frac{x_n-x_{n-1}}{\Delta t}\right)^2\;+\;S^{\text{Lagr}}_{\text{discr}}(x_0,\ldots,x_N)
\end{aligned}\tag{33}$$

where

$$\begin{aligned}
S^{\text{Lagr}}_{\text{discr}}(x_0,\ldots,x_N) &= \Delta t\sum_n\left[\frac{M}{2}\left(\frac{x_n-x_{n-1}}{\Delta t}\right)^2\;-\;V(x_n)\right]\\
&\xrightarrow[N\to\infty]{}\int\!dt\left[\frac{M}{2}\left(\frac{dx}{dt}\right)^2\;-\;V(x)\right]\;=\;S^{\text{Lagr}}[x(t)]
\end{aligned}\tag{34}$$

is the discretized action for of the Lagrangian formalism. In light of eq. (33) we may write the discretized path integral (18) as

$$\begin{aligned}
&\int dx_1 \cdots \int dx_{N-1} \int \frac{dp_1}{2\pi\hbar} \cdots \int \frac{dp_N}{2\pi\hbar} \exp\left(\frac{i}{\hbar}\, S^{\rm Ham}_{\rm discr}(x_0,\ldots,x_N;p_1,\ldots,p_N)\right) \;= \\
&\qquad = \int dx_1 \cdots \int dx_{N-1} \exp\left(\frac{i}{\hbar}\, S^{\rm Lagr}_{\rm discr}(x_0,\ldots,x_N)\right) \times \\
&\qquad\qquad \times \prod_{n=1}^{N} \int \frac{dp_n}{2\pi\hbar} \exp\left(\frac{-i\Delta t}{2M\hbar}\left(p_n \;-\; M\frac{x_n-x_{n-1}}{\Delta t}\right)^2\right)
\end{aligned} \tag{35}$$

where we integrate over all momenta p_n before we integrate over the coordinates. Consequently, in each integral on the last line of eq. (35) we may shift the integration variable from p_n to $p'_n = p_n - M\Delta x_n/\Delta t$, thus

$$\int \frac{dp_n}{2\pi\hbar} \exp\left(\frac{-i\Delta t}{2M\hbar}\left(p_n \;-\; M\frac{x_n-x_{n-1}}{\Delta t}\right)^2\right) \;=\; \int \frac{dp'_n}{2\pi\hbar} \exp\left(\frac{-i\Delta t}{2M\hbar}\, p'^2_n\right) \;=\; \sqrt{\frac{M}{2\pi i\hbar\Delta t}}\,. \tag{36}$$

Plugging this formula back into eq. (35), we arrive at the Lagrangian path integral

$$\begin{aligned}
U(B;A) \;&=\; \lim_{N\to\infty}\left(\frac{MN}{2\pi i\hbar(t_B-t_A)}\right)^{N/2} \times \int dx_1 \cdots \int dx_{N-1} \exp\left(\frac{i}{\hbar}\, S^{\rm Lagr}_{\rm discr}(x_0,\ldots,x_N)\right) \\
&\equiv \iiint\limits_{x(t_A)=x_A}^{x(t_B)=x_B} \mathcal{D}'[x(t)] \exp\left(\frac{i}{\hbar}\, S^{\rm Lagr}[x(t)]\right).
\end{aligned} \tag{37}$$

Note however that in the Lagrangian formalism, the $\mathcal{D}'[x(t)]$ is not just the limit of $d^{N-1}x \equiv dx_1 \cdots dx_{N-1}$ but also includes the normalisation factor

$$C(N,M,t_B-t_A) \;=\; \left(\frac{MN}{2\pi i\hbar(t_B-t_A)}\right)^{N/2}. \tag{38}$$

This normalization factor depends on N, on the net time $T = t_B - t_A$, and on the particle's mass M, but it does not depend on the potential $V(x)$ or the initial and final points x_A and x_B. Consequently, *without discretizing time, a Lagrangian path integral calculation yields the amplitude $U(B;A)$ up to an unknown overall factor $F(M,T)$.* However, we may obtain this factor by comparing with a similar path integral for a free particle: the overall $F(M,T)$ factor is the same in both cases, and the free amplitude is known to be

$$U_{\text{free}}(B;A) \;=\; \sqrt{\frac{M}{2\pi i\hbar T}} \times \exp\left(\frac{iM(x_B - x_A)^2}{2\hbar T}\right). \tag{39}$$

Alternatively, all kind of quantities can be obtained from the ratios of path integrals, and such ratios do not depend on the overall normalization of the $\mathcal{D}[x(t)]$; this is the method most commonly used in the quantum field theory.

Partition Function

The partition function of a quantum system with a Hamiltonian $\hat{H}$ is the trace

$$Z(t) \;\stackrel{\text{def}}{=}\; \operatorname{Tr}\hat{U}(t;0) \;\equiv\; \operatorname{Tr}\exp(-it\hat{H}/\hbar) \;=\; \sum_{\text{eigenvalues } E_n} \exp(-itE_n/\hbar). \tag{40}$$

This time-dependent partition function is related to the temperature-dependent partition function of Statistical Mechanics

$$Z(\beta) \;=\; \operatorname{Tr}\exp(-\beta\hat{H}) \tag{41}$$

via analytical continuation of time t to imaginary values

$$t \;\to\; i\hbar\beta \;=\; \frac{i\hbar}{k_B \times \text{Temperature}}\,. \tag{42}$$

In the path integral formalism, the partition function is given by

$$Z(T) \;=\; \int\! dx\, U(t,x;0,x) \;=\; \iiint\limits_{x(T)=x(0)} \mathcal{D}[x(t)]\, e^{iS[x(t)]/\hbar}. \tag{43}$$

Note no prime over $\mathcal{D}$ because the paths $x(t)$ are subject to only one boundary condition — periodicity in time, $x(T) = x(0)$. Without discretizing time, the path integral (43) can be calculated up to an overall normalization constant. Consequently, when we extract the Hamiltonian's spectrum $\{E_n\}$ from the partition function $Z(T)$, the multiplicity of all the eigenvalues can be determined only up to some unknown overall factor.

For example, consider a harmonic oscillator with action

$$S[x(t)] \;=\; \frac{M}{2}\int\!dt\,\bigl(\dot{x}^2(t)\,-\,\omega^2x^2(t)\bigr)\,. \tag{44}$$

This action is a quadratic functional of the $x(t)$, and it can be diagonalized via Fourier transform,

$$x(t) \;=\; \sum_{n=-\infty}^{+\infty} y_n\times e^{2\pi int/T}, \qquad y_n^* \;=\; y_{-n}\,, \tag{45}$$

$$S[x(t)] \;=\; \sum_{n=-\infty}^{+\infty} C_n y_n^* y_n\,, \tag{46}$$

$$C_n \;=\; C_{-n} \;=\; \frac{MT}{2}\times\left(\left(\frac{2\pi n}{T}\right)^2\,-\,\omega^2\right). \tag{47}$$

Note that the discrete frequencies $2\pi n/T$ of the Fourier transform (45) are completely determined by the boundary conditions $x(T)=x(0)$ and have nothing to do with the oscillator's frequency ω. By linearity of the transform (45),

$$\begin{aligned}\iiint\limits_{\text{periodic}}\mathcal{D}[x(t)] \;&=\; \prod_{n=-\infty}^{+\infty}\int\!dy_n\times\text{a constantJacobian}\\ &=\; J\times\int\!dy_0\prod_{n=1}^{\infty}\int\!d\,\mathrm{Re}\,y_n\!\int\!d\,\mathrm{Im}\,y_n\,.\end{aligned} \tag{48}$$

To be precise, the Jacobian J here depends on T and on the mass M via the normalization of the Lagrangian path integral, but it does not depend on any of the y_n variables, and it does not depend on the oscillator's frequency ω.

In terms of the Fourier variables y_n, the path integral (43) becomes

$$\begin{aligned}Z \;&=\; J\times\int\!dy_0\prod_{n=0}^{\infty}\int\!d\,\mathrm{Re}\,y_n\!\int\!d\,\mathrm{Im}\,y_n\,\exp\left(\frac{i}{\hbar}\,S\;=\;\frac{iC_0}{\hbar}\,y_0^2\,+\,\sum_{n=1}^{\infty}\frac{2iC_n}{\hbar}\,|y_n|^2\right)\\ &=\; J\times\sqrt{\frac{\pi i\hbar}{C_0}}\times\prod_{n=1}^{\infty}\frac{\pi i\hbar}{2C_n}\,.\end{aligned} \tag{49}$$

The coefficients C_n are spelled out in eq. (47), but it's convenient to rewrite them as

$$C_0 \;=\; -\frac{M}{2T}\times(\omega T)^2, \qquad C_{n>0} \;=\; \frac{2\pi^2 M n^2}{T}\times\left(1\,-\,\left(\frac{\omega T}{2\pi n}\right)^2\right). \tag{50}$$

Consequently, the partition function (49) becomes

$$Z(T) \;=\; \frac{F}{(\omega T)\prod\limits_{n=1}^{\infty}\left(1\,-\,\left(\frac{\omega T}{2\pi n}\right)^2\right)} \tag{51}$$

where

$$F \;=\; J\times\sqrt{\frac{2\pi\hbar T}{iM}}\times\prod_{n=1}^{\infty}\frac{i\hbar T}{4\pi^2 M n^2} \tag{52}$$

combines all the factors that do not depend on the oscillator's frequency ω. *A priori,* F could be a function of M or T, but by the non-relativistic dimensional analysis, a dimensionless function $F(M,T,\hbar)$ which does not depend on anything else must be a constant. It is not clear whether this constant is finite or infinite: it contains an infinite product over n that is badly divergent, and the Jacobian J is also badly divergent. To resolve this issue, we need to discretize time and then go through a calculation similar to the above but more complicated; I'll presented in a separate `write up` you should read as a part of your next homework. For now, just take it without proof that all the divergences cancel out and F is finite.

The remaining infinite product in the denominator of eq. (51) is absolutely convergent, and it may be evaluated just by looking at its poles and zeros. The analytic function

$$s(x) \;=\; \frac{1}{x}\times\prod_{n=1}^{\infty}\left(\frac{1}{1-(x/n)^2} \;=\; \frac{n}{n-x}\times\frac{n}{n+x}\right) \tag{53}$$

has no zeroes, it has simple poles at all integers (positive, negative, and zero), it does not have any worse-than-pole singularities in the complex x plane, and it does not grow when $x\to\pm i\infty$. These facts completely determine this function to be

$$\frac{1}{x}\times\prod_{n=1}^{\infty}\frac{1}{1-(x/n)^2} \;=\; \frac{\pi}{\sin(\pi x)} \tag{54}$$

where the normalization comes from the residue of the pole at $x=0$. In eq. (51) we have a similar product for $x=\omega T/2\pi$, hence

$$Z(T) \;=\; \frac{\frac{1}{2}F}{\sin(\omega T/2)}\,. \tag{55}$$

To extract the oscillator's eigenvalues from this partition function, we expand it as

$$Z(T) \;=\; \frac{\frac{1}{2}F}{\sin(\omega T/2)} \;=\; \frac{iF}{e^{i\omega T/2}-e^{-i\omega T/2}} \;=\; iF\times\sum_{n=0}^{\infty} e^{-i\omega T(n+\frac{1}{2})}. \tag{56}$$

Comparing this series to eq. (40), we immediately see that the eigenvalues are $E_n=\hbar\omega(n+\frac{1}{2})$ and they all have the same multiplicity iF. Of course, we all new those facts back in the undergraduate school (if not earlier), but now we know how to derive them in the path-integral formalism.